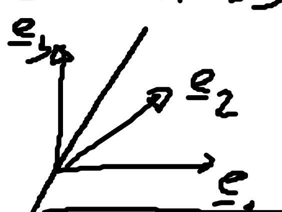
2.4.4 Drehungen/Spiegelungen

- Probleme: (1) Beschreibe Drehungen/Spiegelungen von $\underline{a} \in V$
  (2) Darstellung von $\underline{a}$ bzgl. neuer ONBasis
  → starrer Körper (klass. Mechanik)
  → Symmetriebetrachtungen (Eugene Wigner)
- Zuerst:

ONB (Orthonormalbasis):

$$\{\underline{e}_1, \underline{e}_2, \underline{e}_3\} \xrightarrow[\text{Spiegelung}]{\text{Drehung um bel. Achse}} \{\underline{e}'_1 = \underbrace{D\underline{e}_1}_{\text{gedrehtes } \underline{e}_1}, \underline{e}'_2, \underline{e}'_3\}$$



$$\boxed{\underline{e}'_i = D_{ij}\,\underline{e}_j \quad \text{mit} \quad D_{ij} = \underline{e}'_i \cdot \underline{e}_j = \cos\sphericalangle(\underline{e}'_i, \underline{e}_j)} \quad (2.32)$$

- Eigenschaften der "Drehmatrix" D_{ij} $(\in \mathbb{R}^3 \times \mathbb{R}^3)$

Winkel- und Normerhaltend (Isometrie)

$$\underline{e}'_i \cdot \underline{e}'_j = \delta_{ij} = (D_{ik}\underline{e}_k)\cdot(D_{jl}\underline{e}_l) =$$

$$= D_{ik} D_{jl} \underbrace{\underline{e}_k \cdot \underline{e}_l}_{\delta_{kl}}$$

$$\Longrightarrow \boxed{\begin{aligned} D_{ik} D_{jk} &= \delta_{ij} \\ \underline{\underline{D}}\,\underline{\underline{D}}^t &= \underline{\underline{1}} \\ \rightarrow \quad \underline{\underline{D}}^t &= \underline{\underline{D}}^{-1} \end{aligned}} \quad (2.33)$$

... orthogonale Matrix $\in O(3)$

(nicht kommutative Gruppe aller Drehungen od. Spiegelungen)

Bem 1: Schreibweise: $(\underline{\underline{A}}\,\underline{\underline{B}})_{ik} = A_{ij} B_{jk}$! (2.34)

Matrizenmultiplikation

$$\underbrace{\begin{pmatrix} 1 \text{———} \\ 2 \text{———} \\ 3 \text{———} \end{pmatrix}}_{\underline{\underline{A}}} \underbrace{\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} | & | & | \\ | & | & | \\ | & | & | \end{pmatrix} \end{array}}_{\underline{\underline{B}}} = \underbrace{\begin{pmatrix} A_{1j} B_{j1} & A_{1j} B_{j2} & \cdots \\ & & \\ & & \end{pmatrix}}_{\underline{\underline{A}}\,\underline{\underline{B}}}$$

Bem 2: $\underline{\underline{D}}^{-1}$ ... macht Drehung $\underline{\underline{D}}$ rückgängig = zu $\underline{\underline{D}}$ invers

- „Konstruktion": $\underline{\underline{D}} = \begin{pmatrix} [\underline{e}'_1] \\ [\underline{e}'_2] \\ [\underline{e}'_3] \end{pmatrix}$ — $[\underline{e}'_1 \cdot \underline{e}_1, \underline{e}'_1 \cdot \underline{e}_2, \underline{e}'_1 \cdot \underline{e}_3]$ (2.35)

- Problemlösung:

(1) Drehung / Spiegelung von $\underline{a}$: $\underline{a} \longrightarrow \underline{\underline{D}}\,\underline{a}$, $\underline{a} = a_i \underline{e}_i$

Darstellung in $\{\underline{e}_i\}$: $\underline{\underline{D}}\,\underline{a} = a_i \underline{e}'_i \underset{(2.32)}{=} a_i D_{ij} \underline{e}_j$

$$\longrightarrow \boxed{\begin{aligned} [\underline{\underline{D}}\,\underline{a}]_j &= a_i D_{ij} \\ &= D^t_{ji} a_i \end{aligned}} \quad (2.36)$$

„aktiver" Standpkt

(2) $\underline{a}$ in neuer Basis: $\underline{a} = a_j \underline{e}_j = a'_i \underline{e}'_i$

$\longrightarrow$ Trafo. von Vektorkomp.

$$a'_i = \underline{e}'_i \cdot \underline{a} = \underbrace{\underline{e}'_i \cdot \underline{e}_j}_{D_{ij}} a_j$$

$$\longrightarrow \boxed{\begin{aligned} a'_i &= D_{ij}\, a_j \\ a_i &= D^t_{ij}\, a'_j \end{aligned}} \quad (2.37)$$

NR:

$$a'_i = D_{ij}\, a_j \quad \Big|\, D^{-1}_{ki} \Big|$$

$$D^t_{ki}\, a'_i = \underbrace{D^{-1}_{ki} D_{ij}}_{\delta_{kj}} a_j$$

$k \to i$, $i \to j$

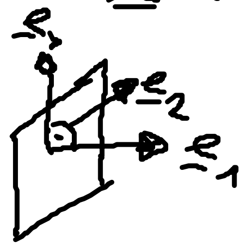
$$D^t_{ki}\, a'_i = a_k$$

"passiver" Standpkt

Schreibweise:

$$\underline{a} \begin{cases} \text{Vektor im Raum} \\ \in \mathbb{R}^n \end{cases} \xrightarrow[(2.37)]{} \underline{a}' = \begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix} \overset{(2.37)}{=} \underline{\underline{D}}\, \underline{a} \quad (2.38)$$

- Bsp. 1: Spiegelung an Ebene $\perp \underline{e}_1$

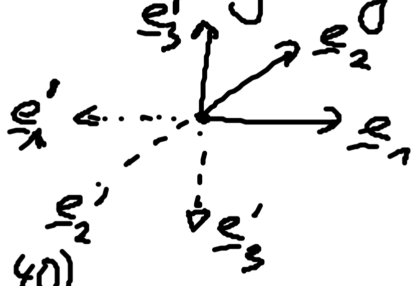


$\longrightarrow$ $\underline{e}'_1 = -\underline{e}_1$, $\underline{e}'_2 = \underline{e}_2$, $\underline{e}'_3 = \underline{e}_3$

$$\longrightarrow \underline{\underline{D}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

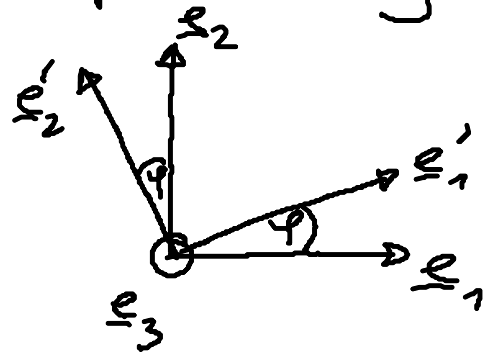
- Bsp 2: "Raumspiegelung" = Pkt. Spiegelung am "Ursprung"

$\longrightarrow$ $\underline{e}'_i = -\underline{e}_i$



$$\longrightarrow \underline{\underline{D}} = \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix} \quad (2.40)$$

- Bsp 3: Drehung um $\underline{e}_3$ (z-Achse):



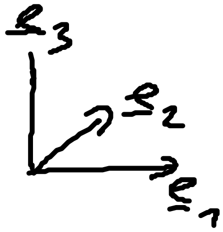
$$\underline{\underline{D}}(\underline{e}_3,\varphi) = [\underline{e}_i' \cdot \underline{e}_j] = \begin{array}{c} \underline{e}_1' \text{ in } \{\underline{e}_1\} \\ \begin{pmatrix} \boxed{\cos\varphi \quad \sin\varphi \quad 0} \\ -\sin\varphi \quad \cos\varphi \quad 0 \\ 0 \quad 0 \quad 1 \end{pmatrix} \end{array} \tag{2.41}$$

- Bsp 4: <u>Eulersche Winkel</u> (verschiedene Konv.)

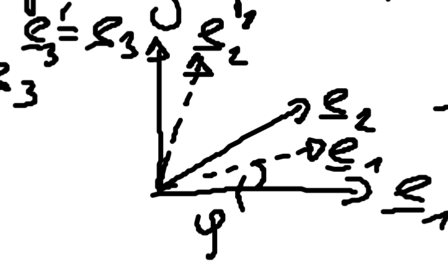
⟶ allg. Orientierung von $\underline{e}' = D\,\underline{e}_i$

⟶ starrer Körper klass. Mechanik

(⟶ Computergraphik)



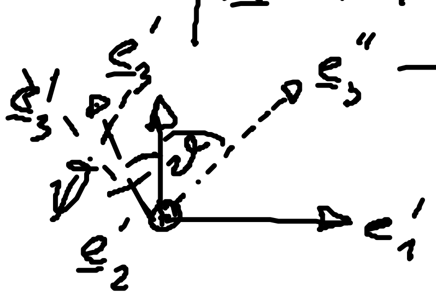
$\xrightarrow[\text{um } \varphi]{\text{Rotation um } \underline{e}_3}$



$\xrightarrow[\text{um } \vartheta]{\text{um } \underline{e}_2'}$ $\underline{e}_1'', \underline{e}_2'', \underline{e}_3''$

$\xrightarrow[\text{um } \psi]{\text{um } \underline{e}_3''}$ $\underline{e}_1''', \underline{e}_2''', \underline{e}_3'''$:

$$\boxed{\underline{\underline{D}}(\varphi,\vartheta,\psi) \underset{(2.41)}{=} \underbrace{\underline{\underline{D}}(\underline{e}_3'',\psi)}_{\text{in } \{\underline{e}_i''\}} \; \underbrace{\underline{\underline{D}}(\underline{e}_2',-\vartheta)}_{\text{in } \{\underline{e}_i'\}} \; \underline{\underline{D}}(\underline{e}_3,\varphi)} \tag{2.42}$$



⟶ allg. Drehung/Spiegelung: 3 Winkel