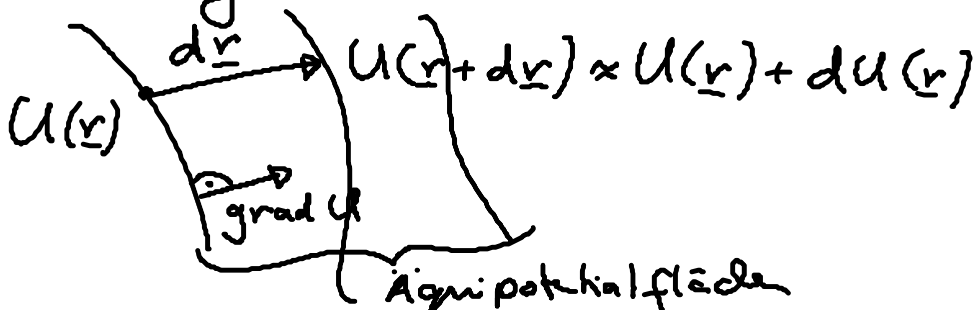
6.4. Der Nabla-Operator

- Erinnerung:



$$dU(\underline{r}) = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 + \frac{\partial U}{\partial x_3} dx_3 \quad (6.19)$$

$$dU(\underline{r}) = \operatorname{grad} U \cdot d\underline{r} \quad (6.20)$$

$$\text{mit } d\underline{r} = \frac{\partial \underline{r}}{\partial x_i} dx_i = \left|\frac{\partial \underline{r}}{\partial x_i}\right| \underline{e}_i \, dx_i \quad (6.21)$$

$$\Longrightarrow \boxed{\operatorname{grad} U = \frac{1}{\left|\frac{\partial \underline{r}}{\partial x_i}\right|} \frac{\partial U}{\partial x_i} \underline{e}_i \quad (\text{für } \underline{e}_i \cdot \underline{e}_j = \delta_{ij})} \quad (6.22)$$

- Nabla-Operator: $$\boxed{\underline{\nabla} = \underline{e}_i \frac{1}{\left|\frac{\partial \underline{r}}{\partial x_i}\right|} \frac{\partial}{\partial x_i} \quad \text{so daß: } \operatorname{grad} U = \underline{\nabla} U} \quad (6.23)$$

- entlang Äquipotentialflächen

  $dU = 0 \xrightarrow{(6.20)} \operatorname{grad} U \perp d\underline{r}$

  $\rightarrow$ grad U $\parallel$ Richtung maximaler Änderung von U

- Koordinatensysteme:

  a) kartesische Koordinaten

$$d\underline{r} = dx\,\underline{e}_x + dy\,\underline{e}_y + dz\,\underline{e}_z$$

$$(6.22) \longrightarrow \boxed{\begin{aligned} \operatorname{grad} U &= \underline{e}_x \frac{\partial U}{\partial x} + \underline{e}_y \frac{\partial U}{\partial y} + \underline{e}_z \frac{\partial U}{\partial z} \\ \underline{\nabla} &= \underline{e}_x \frac{\partial}{\partial x} + \underline{e}_y \frac{\partial}{\partial y} + \underline{e}_z \frac{\partial}{\partial z} \end{aligned}} \quad (6.24)$$

Bsp: $U \sim r^2 = x^2 + y^2 + z^2 \longrightarrow \underline{\nabla} U \sim 2x\,\underline{e}_x + 2y\,\underline{e}_y + 2z\,\underline{e}_z$
$= 2\underline{r}$

b) Zylinderkoordinaten (6.25)

$$\left.\begin{aligned} d\underline{r} &\overset{(5.9)}{=} d\varrho\,\underline{e}_\varrho + \varrho\, d\varphi\,\underline{e}_\varphi + dz\,\underline{e}_z \\ dU &= \frac{\partial U}{\partial \varrho} d\varrho + \frac{\partial U}{\partial \varphi} d\varphi + \frac{\partial U}{\partial z} dz \end{aligned}\right\} \quad \boxed{\begin{aligned} \underline{\nabla} = \underline{e}_\varrho \frac{\partial}{\partial \varrho} + \underline{e}_\varphi \frac{1}{\varrho}\frac{\partial}{\partial \varphi} \\ + \underline{e}_z \frac{\partial}{\partial z} \end{aligned}}$$

Bsp: $U(\varrho) \sim \ln \varrho \longrightarrow \nabla U \sim \frac{1}{\varrho}\underline{e}_\varrho$

c) Kugelkoordinaten:

$$\boxed{\underline{\nabla} = \underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\vartheta \frac{1}{r}\frac{\partial}{\partial \vartheta} + \underline{e}_\varphi \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \varphi}} \quad (6.26)$$

Beweis: Übungen

- Rechenregeln:

$$\left.\begin{aligned} \text{(i)}\quad \underline{\nabla}(cU) &= c\,\nabla U \qquad , c \in \mathbb{R} \\ \underline{\nabla}(U+V) &= \underline{\nabla} U + \underline{\nabla} V \\ \underline{\nabla}(UV) &= (\nabla U)V + U(\underline{\nabla} V) \end{aligned}\right\} (6.27)$$

(ii) $\underline{\nabla}(\underline{a}\cdot\underline{r}) = \underline{a}$ $\qquad$ $\underline{a}$ ... konst. Vektor

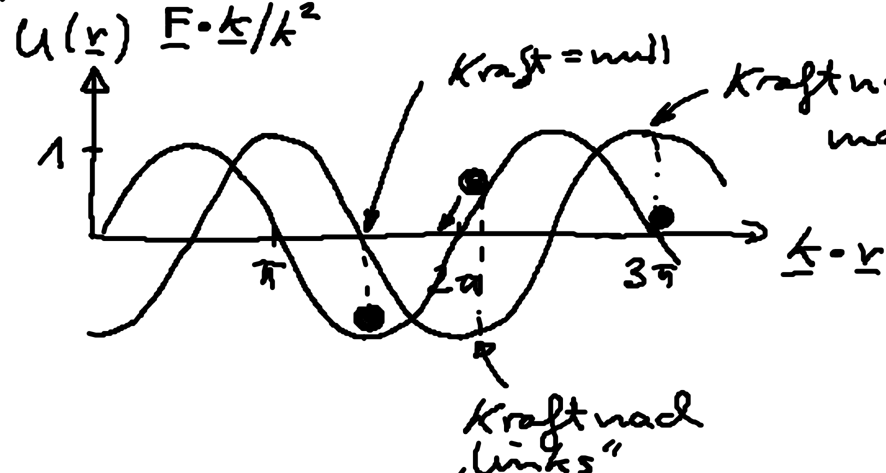
$$\left.\begin{aligned} &\underline{\nabla}(\underline{a}\cdot\underline{r}) = \underline{a} \\ &\underline{\nabla}\, r = \hat{\underline{r}}\,, \quad \underline{r} = r\,\hat{\underline{r}} = r\,\underline{e}_r \\ &\underline{\nabla} f(r) = \frac{\partial f}{\partial r}\,\hat{\underline{r}} \\ &\text{insbesondere: } \nabla\frac{1}{r} = -\frac{\hat{\underline{r}}}{r^2} = -\frac{\underline{r}}{r^3} \end{aligned}\right\} \quad (6.28)$$

· Richtungsableitung: „Ableitung entlang $\hat{\underline{\nu}}$", $|\hat{\underline{\nu}}|=1$

<u>Def</u>: $\boxed{\begin{aligned} &\text{Richtungsableitung } \hat{\underline{\nu}}\cdot\underline{\nabla}U \qquad (6.29)\\ &\text{so daß: mit } d\underline{r} = \hat{\underline{\nu}}\,ds:\ dU = (\hat{\underline{\nu}}\cdot\nabla U)\,ds \end{aligned}}$

· <u>Beispiele</u>: $U(\underline{r})$ ... potentielle Energie
$\longrightarrow \underline{F}(\underline{r}) = -\underline{\nabla}U(\underline{r})$ ... Kraftfeld $\Big\}$ s. Mechanik (6.30)

(i) $U(\underline{r}) \sim \sin \underline{k}\cdot\underline{r} \longrightarrow \underline{F}(\underline{r}) \sim -\underline{k}\cos(\underline{k}\cdot\underline{r})$ (6.31)



(ii) $U(r) \sim \frac{1}{r}$ ... kugelsymmetr. / Zentral-Potential

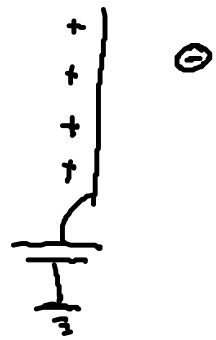
$\xrightarrow[\text{Kugel-koord.}]{(6.28)}$ $\underline{F}(\underline{r}) \sim \frac{\underline{e}_r}{r^2}$ ... Kraft ∥ radialer Richtg.

(6.32) ... Grav. kraft im Schwerefeld eines Planeten

(iii) $U(\varrho) \sim \ln\varrho$ ... zylindersymmetr. Potential

$\xrightarrow[\text{Koord. (6.25)}]{\text{Zyl.}} \quad F(\underline{r}) \sim -\frac{1}{\varrho}\,\underline{e}_\varrho \qquad (6.33)$

(vgl. Kap 6.1 & 6.2)



6.5 Divergenz

- Erinnerung: $\operatorname{grad} U(\underline{r}) = \underline{\nabla}\, U(\underline{r})$ ... Vektor

  $\longrightarrow$ weitere Operationen von $\underline{\nabla}$?

- **Def:**

$$\boxed{\begin{array}{l}\text{Divergenz eines Vektorfeldes } \underline{a}(\underline{r}) \\ \qquad \operatorname{div} \underline{a}(\underline{r}) = \underline{\nabla} \cdot \underline{a}(\underline{r}) \\ \text{... Quellenfeld von } \underline{a}(\underline{r}) \quad \text{(Skalarfeld)}\end{array}}$$

- kartesische Koord:

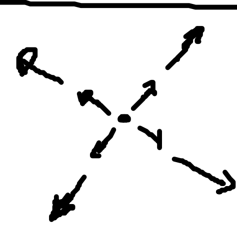
$$\underline{\nabla} \overset{(6.24)}{=} \underline{e}_i \frac{\partial}{\partial x_i}, \qquad \underline{a}(\underline{r}) = a_i(\underline{r})\, \underline{e}_i \qquad \begin{array}{l} i = x, y, z \\ x_i = x, y, z \end{array}$$

$$\longrightarrow \underline{\nabla} \cdot \underline{a}(\underline{r}) = \left(\underline{e}_i \frac{\partial}{\partial x_i}\right) \cdot \left(a_j(\underline{r})\, \underline{e}_j\right)$$

$$\left[\frac{\partial}{\partial x_i} \underline{e}_j = 0\right] = \left(\frac{\partial}{\partial x_i} a_j\right) \underbrace{\underline{e}_i \cdot \underline{e}_j}_{\delta_{ij}}$$

$$\boxed{\underline{\nabla} \cdot \underline{a}(\underline{r}) = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z} \qquad (6.35)$$

- Bsp 1: $\underline{a}(\underline{r}) = \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$



(6.36)

$\longrightarrow \underline{\nabla} \cdot \underline{r} = 1 + 1 + 1 = 3$

Bsp 2: Kugelsymmetr. Quellen/Senkenfeld:

$$\underline{a}(\underline{r}) = a(r)\,\underline{e}_r = \frac{a(r)}{r}\,\underline{r} \quad , \quad \underline{e}_r = \frac{\underline{r}}{r} \qquad (6.37)$$

<u>o.B:</u>
$$\underline{\nabla}\cdot\underline{a}(\underline{r}) = 3\,\frac{a(r)}{r} + \frac{\partial\, a(r)/r}{\partial r}\, r = 2\,\frac{a(r)}{r} + \frac{\partial a}{\partial r} \qquad (6.38)$$

für $a(r) = \frac{1}{r^2}$: $\nabla\cdot\underline{a}(\underline{r}) = 0\,!$ für $r \neq 0$

(6.39) ($r=0$ .. Singularität)

Punktmasse/-ladung
bei $r=0$ erzeugt Feld,
sonst keine Quelle

(6.40)

• <u>Deutung</u>:
$$\boxed{\underline{\nabla}\cdot\underline{a}(\underline{r}) \text{ identifiziert lokal Quellen \& Senken von Vektorfeldern = "Flüssen"}}$$

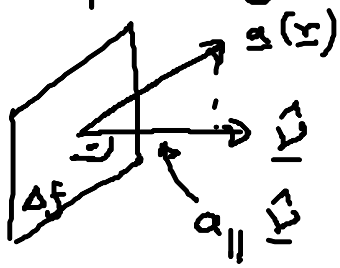
Bsp: $\underline{a}(\underline{r}) = \underline{v}(\underline{r})$ ... Geschw. feld in einer Flüssigkeit

aber auch $\underline{E}(\underline{r})$, <u>nicht</u> $\underline{B}(\underline{r})$ [div $\underline{B}(\underline{r}) = 0$!]

keine magnet. Monopole

<u>Motivation</u>

(1) Fluß durch Fläche mit Normalen $\hat{\underline{\nu}}$ ($|\hat{\nu}| = 1$)
und Größe Δf



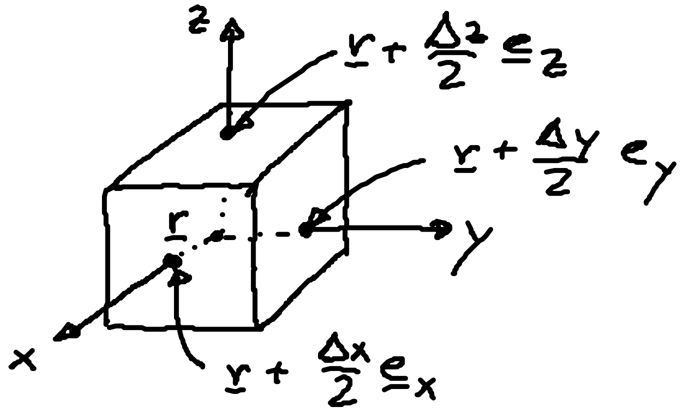
$a_\parallel \Delta f = (\underline{a}\cdot\hat{\underline{\nu}})\,\Delta f$
tritt durch die Fläche

$\underline{a}_\perp \perp \hat{\nu}$ nicht!

insbesondere: $\hat{\underline{\nu}} = \pm\,\underline{e}_x \longrightarrow a_\parallel = \pm\, a_x$

(2) Fluß aus kleinem Vol.element $\Delta V = \Delta x \Delta y \Delta z$ um $\underline{r}$

Vol. eines Quaders



Aufgabe: Summe über alle Flüsse aus ΔV hinaus!

Konvention: $\boxed{\hat{\underline{\nu}} \text{ zeigt immer aus } \Delta V \text{ hinaus}}$ (6.41)

$$\longrightarrow \underline{a} \cdot \hat{\underline{\nu}} > 0, \text{ für Quelle!}$$

hier:

$$q_x^{\pm} = \pm a_x\left(\underline{r} \pm \frac{\Delta x}{2} \underline{e}_x\right) \overset{\text{Taylor}}{=\!=} \pm\left[a_x(\underline{r}) \pm \frac{\Delta x}{2} \frac{\partial}{\partial x} a_x\right]$$

$\hat{\nu} = +\underline{e}_x$, $\hat{\nu} = -\underline{e}_x$

$$q_y^{\pm} = \pm a_y\left(\underline{r} \pm \frac{\Delta y}{2} \underline{e}_y\right) = \pm\left[a_y(\underline{r}) \pm \frac{\Delta y}{2} \frac{\partial}{\partial y} a_y\right]$$

$$q_z^{\pm} = \ldots$$

$\longrightarrow$ Fluß aus ΔV:

$$\begin{aligned} q(\underline{r})\, \Delta V &= (q_x^+ + q_x^-)\, \Delta y \Delta z \\ &\quad + (q_y^+ + q_y^-)\, \Delta x \Delta z \\ &\quad + \ldots \\ &= \left\{\left[\cancel{a_x(\underline{r})} + \frac{\Delta x}{2} \frac{\partial}{\partial x} a_x\right] - \left[\cancel{a_x(\underline{r})} - \frac{\Delta x}{2} \frac{\partial}{\partial x} a_x\right]\right\} \Delta y \Delta z \end{aligned}$$

$$+ \dots \Delta x \Delta z \quad + \dots \Delta x \Delta y$$

$$= \left(\frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z \right) \underbrace{\Delta x \Delta y \Delta z}_{\Delta U}$$

$\longrightarrow$ $\boxed{q(\underline{r})\,\Delta U \text{ mit } q(\underline{r}) = \operatorname{div} \underline{a}(\underline{r}) = \underline{\nabla} \cdot \underline{a}(\underline{r})}$ (6.43)

mißt Fluß aus ΔU heraus