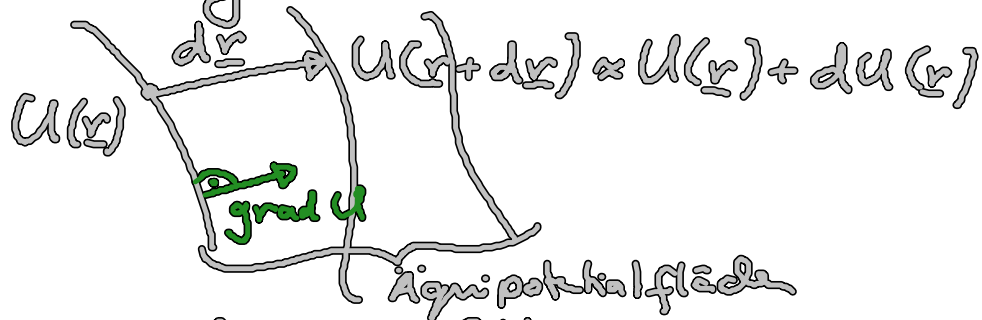


6.4. Der Nabla-Operator

- Erinnerung:



Aquipotentialfläche

$$dU(\underline{r}) = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 + \frac{\partial U}{\partial x_3} dx_3 \quad (6.19)$$

$$dU(\underline{r}) = \text{grad } U \cdot d\underline{r} \quad (6.20)$$

$$\text{mit } d\underline{r} = \frac{\partial \underline{r}}{\partial x_i} dx_i = \left| \frac{\partial \underline{r}}{\partial x_i} \right| \underline{e}_i dx_i \quad (6.21)$$

\Rightarrow

$$\text{grad } U = \frac{1}{\left| \frac{\partial \underline{r}}{\partial x_i} \right|} \frac{\partial U}{\partial x_i} \underline{e}_i \quad (6.22)$$

(für $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$)

• Nabla-Operator: $\underline{\nabla} = \underline{e}_i \frac{1}{\left| \frac{\partial \underline{r}}{\partial x_i} \right|} \frac{\partial}{\partial x_i} \quad (6.23)$

so daß: $\text{grad } U = \underline{\nabla} U$

• entlang Aquipotentialflächen

$$dU = 0 \quad (6.20) \quad \text{grad } U \perp d\underline{r}$$

$\rightarrow \text{grad } U \parallel$ Richtung maximaler Änderung von U

• Koordinatensysteme:

a) Cartesische Koordinaten

$$dr = dx \underline{e}_x + dy \underline{e}_y + dz \underline{e}_z$$

(6.22)

$$\begin{aligned} \text{grad } U &= \underline{e}_x \frac{\partial U}{\partial x} + \underline{e}_y \frac{\partial U}{\partial y} + \underline{e}_z \frac{\partial U}{\partial z} \\ \underline{\nabla} &= \underline{e}_x \frac{\partial}{\partial x} + \underline{e}_y \frac{\partial}{\partial y} + \underline{e}_z \frac{\partial}{\partial z} \end{aligned} \quad (6.24)$$

Bsp: $U \sim r^2 = x^2 + y^2 + z^2 \longrightarrow \underline{\nabla} U \sim 2x \underline{e}_x + 2y \underline{e}_y + 2z \underline{e}_z = 2 \underline{r}$

b) Zylinderkoordinaten

(6.25)

$$\begin{aligned} dr &= ds \underline{e}_s + s d\varphi \underline{e}_\varphi + dz \underline{e}_z \\ dU &= \frac{\partial U}{\partial s} ds + \frac{\partial U}{\partial \varphi} d\varphi + \frac{\partial U}{\partial z} dz \end{aligned}$$

$$\underline{\nabla} = \underline{e}_s \frac{\partial}{\partial s} + \underline{e}_\varphi \frac{1}{s} \frac{\partial}{\partial \varphi} + \underline{e}_z \frac{\partial}{\partial z}$$

Bsp: $U(s) \sim \ln s \longrightarrow \underline{\nabla} U \sim \frac{1}{s} \underline{e}_s$

c) Kugelkoordinaten:

$$\underline{\nabla} = \underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\vartheta \frac{1}{r} \frac{\partial}{\partial \vartheta} + \underline{e}_\varphi \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \varphi} \quad (6.26)$$

Beweis: Übungen

• Rechenregeln:

$$\begin{aligned} (i) \quad \underline{\nabla}(cU) &= c \underline{\nabla} U, \quad c \in \mathbb{R} \\ \underline{\nabla}(U+V) &= \underline{\nabla} U + \underline{\nabla} V \\ \underline{\nabla}(UV) &= (\underline{\nabla} U)V + U(\underline{\nabla} V) \end{aligned} \quad (6.27)$$

$$\begin{aligned}
 \text{(ii)} \quad \underline{\nabla}(\underline{a} \cdot \underline{r}) &= \underline{a} & \underline{a} \dots \text{ konst. Vektor} \\
 \underline{\nabla} r &= \hat{\underline{r}}, \quad \underline{r} = r \hat{\underline{r}} = r \underline{e}_r \\
 \underline{\nabla} f(r) &= \frac{\partial f}{\partial r} \hat{\underline{r}} \\
 \text{insbesondere: } \underline{\nabla} \frac{1}{r} &= -\frac{\hat{\underline{r}}}{r^2} = -\frac{\underline{r}}{r^3}
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} \underline{\nabla}(\underline{a} \cdot \underline{r}) &= \underline{a} \\ \underline{\nabla} r &= \hat{\underline{r}} \\ \underline{\nabla} f(r) &= \frac{\partial f}{\partial r} \hat{\underline{r}} \end{aligned}} \right\} (6.28)$$

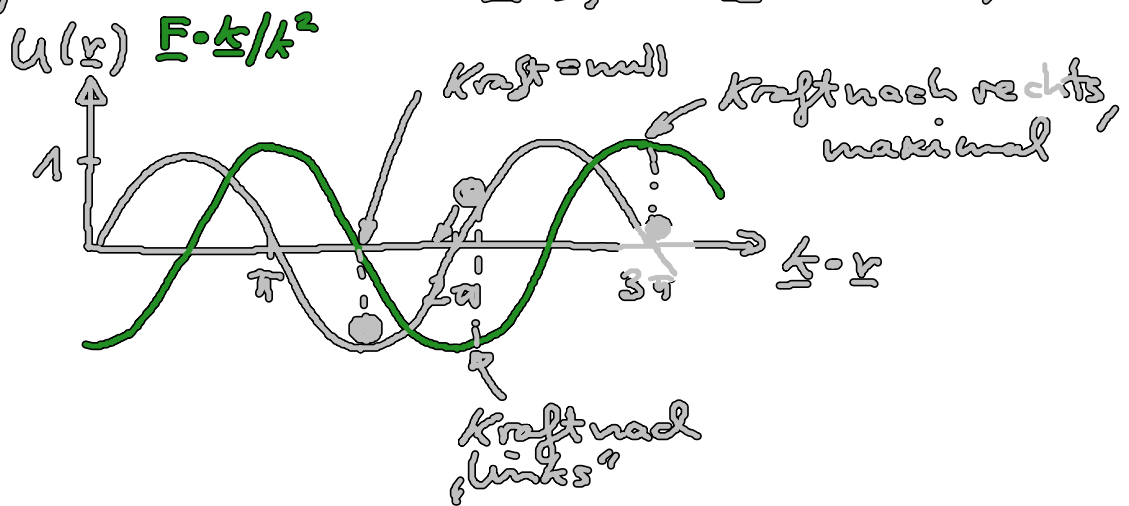
• Richtungsableitung: „Ableitung entlang $\hat{\underline{v}}$ “, $|\hat{\underline{v}}|=1$

Def: Richtungsableitung $\hat{\underline{v}} \cdot \underline{\nabla} U$ (6.29)

so daß: mit $d\underline{r} = \hat{\underline{v}} ds$: $dU = (\hat{\underline{v}} \cdot \underline{\nabla} U) ds$

Beispiele: $U(\underline{r}) \dots$ potentielle Energie $\left. \vphantom{U(\underline{r})} \right\}$ s. Mechanik
 $\longrightarrow \underline{F}(\underline{r}) = -\underline{\nabla} U(\underline{r}) \dots$ Kraftfeld $\left. \vphantom{\underline{F}(\underline{r})} \right\} (6.30)$

(i) $U(\underline{r}) \sim \sin \underline{k} \cdot \underline{r} \longrightarrow \underline{F}(\underline{r}) \sim -\underline{k} \cos(\underline{k} \cdot \underline{r})$ (6.31)



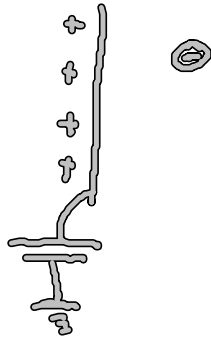
(ii) $U(r) \sim \frac{1}{r} \dots$ kugel-symmetr./zentral-Potential

$\xrightarrow{\text{Kugel-symmetr.}} (6.28) \underline{F}(\underline{r}) \sim \frac{\underline{e}_r}{r^2} \dots$ Kraft \parallel radiale Richtung
 (6.32) \dots Grav.kraft im Schwerfeld eines Planeten

(iii) $U(\underline{s}) \sim \ln s \dots$ zylinder-symmetr. Potential

$$\frac{\text{zyl.}}{\text{sym.}} \rightarrow F(\underline{r}) \sim -\frac{1}{\epsilon_0} \underline{e}_r \quad (6.33)$$

(zyl. Kap
6.1 & 6.2)



6.5 Divergenz

- Erinnerung: $\text{grad } U(\underline{r}) = \underline{\nabla} U(\underline{r})$... Vektor
→ weitere Operationen von $\underline{\nabla}$?

- Def: Divergenz eines Vektorfeldes $\underline{a}(\underline{r})$
 $\text{div } \underline{a}(\underline{r}) = \underline{\nabla} \cdot \underline{a}(\underline{r})$
 ... Quellenfeld von $\underline{a}(\underline{r})$ (Skalarfeld)

- Kartesische Koord:

$$\underline{\nabla} \stackrel{(6.26)}{=} \underline{e}_i \frac{\partial}{\partial x_i}, \quad \underline{a}(\underline{r}) = a_i(\underline{r}) \underline{e}_i \quad \begin{array}{l} i = x, y, z \\ x_i = x, y, z \end{array}$$

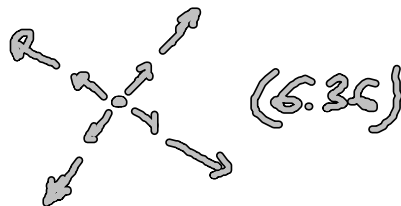
$$\rightarrow \underline{\nabla} \cdot \underline{a}(\underline{r}) = \left(\underline{e}_i \frac{\partial}{\partial x_i} \right) \cdot \left(a_j(\underline{r}) \underline{e}_j \right)$$

$$\left[\frac{\partial}{\partial x_i} \underline{e}_j \cdot \underline{e}_j = \delta_{ij} \right] = \left(\frac{\partial}{\partial x_i} a_j \right) \underbrace{\underline{e}_i \cdot \underline{e}_j}_{\delta_{ij}}$$

$$\underline{\nabla} \cdot \underline{a}(\underline{r}) = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z \quad (6.35)$$

- Bsp 1: $\underline{a}(\underline{r}) = \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\rightarrow \underline{\nabla} \cdot \underline{r} = 1 + 1 + 1 = 3$$



Bsp 2: Kugelsymmetr. Quellen/Senkenfeld:

$$\underline{a}(\underline{r}) = a(r) \underline{e}_r = \frac{a(r)}{r} \underline{r}, \quad \underline{e}_r = \frac{\underline{r}}{r} \quad (6.37)$$

o.B.: $\underline{\nabla} \cdot \underline{a}(\underline{r}) = 3 \frac{a(r)}{r} + \frac{\partial a(r)}{\partial r} r$ (6.38)

$$= 2 \frac{a(r)}{r} + \frac{\partial a}{\partial r}$$

für $a(r) = \frac{1}{r^2}$: $\underline{\nabla} \cdot \underline{a}(\underline{r}) = 0!$ für $r \neq 0$ (6.39) ($r=0$.. Singularität)

Punktmasse/-ladung
bei $r=0$ erzeugt Feld,
sonst keine Quelle

(6.40)

Bedeutung:

$\underline{\nabla} \cdot \underline{a}(\underline{r})$ identifiziert lokal Quellen & Senken von Vektorfeldern = "Flüssen"

Bsp: $\underline{a}(\underline{r}) = \underline{v}(\underline{r})$.. Geschw. feld in einer Flüssigkeit

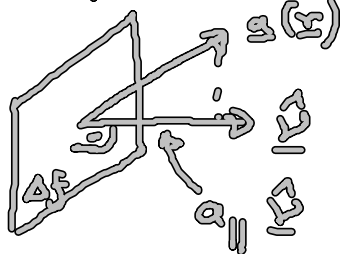
aber auch $\underline{E}(\underline{r})$, nicht $\underline{B}(\underline{r})$ [div $\underline{B}(\underline{r})=0!$]

keine magnet. Monopole

Motivation

(1) Fluß durch Fläche mit Normalen $\hat{\underline{v}}$ ($|\hat{\underline{v}}|=1$)

und Größe Δf



$$a_{\parallel} \Delta f = (\underline{a} \cdot \hat{\underline{v}}) \Delta f$$

tritt durch die Fläche

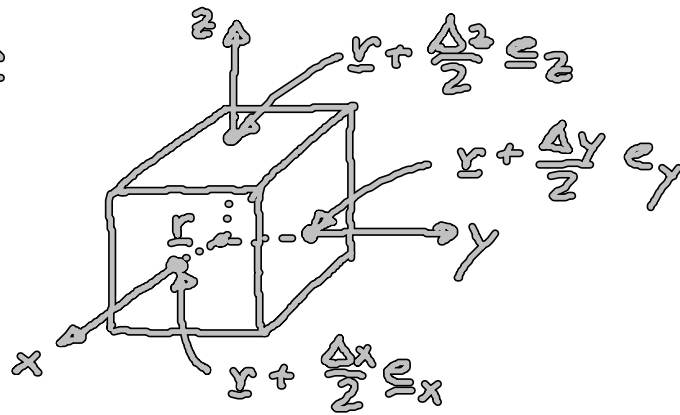
$$a_{\perp} \Delta f \text{ nicht!}$$

insbesondere: $\hat{\underline{v}} = \pm \underline{e}_x \rightarrow a_{\parallel} = \pm a_x$

(2) Fluß aus kleinem Vol. element
um \underline{r}

$$\Delta V = \Delta x \Delta y \Delta z$$

Vol. eines Quaders



Aufgabe: Summe über alle Flüsse aus ΔV hinaus!

Konvention: $\hat{\underline{v}}$ zeigt immer aus ΔV hinaus (6.41)

$$\rightarrow \underline{a} \cdot \hat{\underline{v}} > 0, \text{ für Quelle!}$$

hier:

$$q_x^\pm = \pm \underline{a}_x \left(\underline{r} \pm \frac{\Delta x}{2} \underline{e}_x \right) \stackrel{\text{Taylor}}{=} \pm \left[a_x(\underline{r}) \pm \frac{\Delta x}{2} \frac{\partial}{\partial x} a_x \right]$$

$\hat{\underline{v}} = +\underline{e}_x$ (outward), $\hat{\underline{v}} = -\underline{e}_x$ (inward)

$$q_y^\pm = \pm a_y \left(\underline{r} \pm \frac{\Delta y}{2} \underline{e}_y \right) = \pm \left[a_y(\underline{r}) \pm \frac{\Delta y}{2} \frac{\partial}{\partial y} a_y \right]$$

$$q_z^\pm = \dots$$

\rightarrow Fluß aus ΔV :

$$q(\underline{r}) \Delta V = (q_x^+ + q_x^-) \Delta y \Delta z$$

$$+ (q_y^+ + q_y^-) \Delta x \Delta z$$

$$+ \dots$$

$$= \left\{ \left[\cancel{a_x(\underline{r})} + \frac{\Delta x}{2} \frac{\partial}{\partial x} a_x \right] - \left[\cancel{a_x(\underline{r})} - \frac{\Delta x}{2} \frac{\partial}{\partial x} a_x \right] \right\} \Delta y \Delta z$$

$$+ \dots \Delta x \Delta z \quad + \dots \Delta x \Delta y$$

$$= \left(\frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z \right) \underbrace{\Delta x \Delta y \Delta z}_{\Delta V}$$



$q(\underline{r}) \Delta V$ mit $q(\underline{r}) = \operatorname{div} \underline{a}(\underline{r}) = \nabla \cdot \underline{a}(\underline{r})$ (G.43)
 mißt Fluß aus ΔV heraus