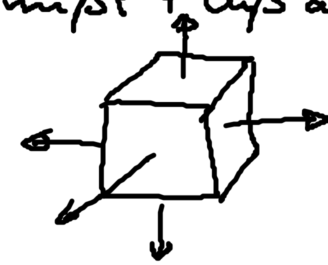
• Erinnerung:

$$\mathrm{div}\,\underline{a}(\underline{r}) = \underline{\nabla}\cdot\underline{a}(\underline{r})$$

kartesische Koordinaten: $\mathrm{div}\,\underline{a} = \frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z$

• Fluß = Normalkomponente von $\underline{a}$ mal Fläche (Normale immer aus Volumen heraus!)

• $\boxed{q(\underline{r}) = \mathrm{div}\,\underline{a}\;\Delta V \text{ mißt Fluß aus } \Delta V \text{ heraus}}$ (6.43)



$$q(\underline{r}) = 0 \;\dots\; \text{"was rein fließt, fließt raus"}$$
$$> 0 \;\dots\; \text{"es fließt mehr raus als rein"} \;\hat{=}\; \text{Quelle}$$
$$< 0 \;\dots\; \text{Senke}$$

Bsp: $\underline{v}(\underline{r}) = v_0\,\underline{r} \longrightarrow \mathrm{div}\,\underline{v} = 3v_0 \longrightarrow$ überall Quellen! (6.44)

• Regeln:

$$\left.\begin{aligned}
&(1)\quad \underline{\nabla}\cdot(\underline{a}+\underline{b}) = \underline{\nabla}\cdot\underline{a} + \underline{\nabla}\cdot\underline{b}\\
&(2)\quad \underline{\nabla}\cdot(f(\underline{r})\underline{a}) = f(\underline{r})\,\underline{\nabla}\cdot\underline{a} + \underline{a}\cdot\underline{\nabla}f(\underline{r})
\end{aligned}\right\}\ (6.45)$$

Beweis: in kartesischen Koord.

• Zylinderkoordinaten:

$$\underline{\nabla}\cdot\underline{a}(\underline{r}) = \left(\underline{e}_\rho\frac{\partial}{\partial\rho} + \underline{e}_\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \underline{e}_z\frac{\partial}{\partial z}\right)\cdot\left(a_\rho\underline{e}_\rho + a_\varphi\underline{e}_\varphi + a_z\underline{e}_z\right)$$

Achtung:

$$\left.\begin{aligned}
&\frac{\partial}{\partial\varphi}\underline{e}_\rho = \frac{\partial}{\partial\varphi}\begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{pmatrix} = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix} = \underline{e}_\varphi\\
&\frac{\partial}{\partial\varphi}\underline{e}_\varphi = -\underline{e}_\rho \quad \text{... nicht benötigt}\\
&\frac{\partial}{\partial x_i}\underline{e}_j = 0 \quad , \quad x_i,\ i = \rho,\varphi,z
\end{aligned}\right\} \quad (6.46)$$

(6.46) und $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$

$$\longrightarrow \boxed{\begin{aligned}\underline{\nabla}\cdot\underline{a}(\underline{r}) &= \frac{\partial}{\partial\rho}a_\rho + \frac{1}{\rho}a_\rho + \frac{1}{\rho}\frac{\partial}{\partial\varphi}a_\varphi + \frac{\partial}{\partial z}a_z\\ &= \frac{1}{\rho}\frac{\partial(\rho a_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial}{\partial\varphi}a_\varphi + \frac{\partial}{\partial z}a_z\end{aligned}} \quad (6.47)$$

Bsp:

- $E(\underline{r}) \sim -\frac{1}{\rho}\underline{e}_\rho \longrightarrow \operatorname{div}\underline{E}(\underline{r}) \sim \frac{1}{\rho^2} - \frac{1}{\rho^2} = 0,\ \rho \neq 0$
- $\underline{v}(\underline{r}) = \omega\rho\,\underline{e}_\varphi \longrightarrow \operatorname{div}\underline{v} = 0!$

- Kugelkoordinaten: $\underline{a} = a_r\underline{e}_r + a_\vartheta\underline{e}_\vartheta + a_\varphi\underline{e}_\varphi$ (6.48)

$$\boxed{\begin{aligned}\underline{\nabla}\cdot\underline{a}(\underline{r}) &= \frac{\partial a_r}{\partial r} + \frac{2}{r}a_r + \frac{1}{r}\frac{\partial a_\vartheta}{\partial\vartheta} + \frac{1}{r\tan\vartheta}a_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial a_\varphi}{\partial\varphi}\\ &= \frac{1}{r^2}\frac{\partial(r^2 a_r)}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial(\sin\vartheta\, a_\vartheta)}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial a_\varphi}{\partial\varphi}\end{aligned}}$$

Beweis: Übungen

Bsp: $\underline{a}(\underline{r}) = \underline{r} = r\,\underline{e}_r \longrightarrow \operatorname{div}\underline{a} = \frac{\partial r}{\partial r} + \frac{2}{r}r = 3!$
(vgl. 6.36)

6.6 Rotation

· **Def**: Rotation eines Vektorfeldes $\underline{a}(\underline{r})$

$$\text{rot}\,\underline{a}(\underline{r}) = \underline{\nabla} \times \underline{a}(\underline{r}) \qquad (6.49)$$

... Vektorfeld = Wirbelfeld von $\underline{a}(\underline{r})$

· kartesische Koord.:

$$\underline{\nabla} = \underline{e}_i \frac{\partial}{\partial x_i}, \quad \underline{a}(\underline{r}) = a_i \underline{e}_i, \quad i, x_i = x, y, z$$

$$\longrightarrow \quad \underline{\nabla} \times \underline{a}(\underline{r}) = \left(\underline{e}_i \frac{\partial}{\partial x_i}\right) \times \left(a_j(\underline{r})\, \underline{e}_j\right)$$

$$\text{mit } \underline{e}_i \times \underline{e}_j = \varepsilon_{ijk}\, \underline{e}_k \quad (2.23)$$

$$\left[\frac{\partial}{\partial x_i} \underline{e}_j = 0\right] \quad = \frac{\partial}{\partial x_i} a_j(\underline{r}) \underbrace{\varepsilon_{ijk}}_{\varepsilon_{kij}} \underline{e}_k$$

ε_{312}

ε_{123}

$(\varepsilon_{213}$

$(\varepsilon_{312} = 1$

$$\xrightarrow[k \to i]{i \to j,\ j \to k}$$

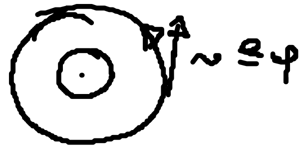
$$\boxed{\underline{\nabla} \times \underline{a}(\underline{r}) = \varepsilon_{ijk} \left(\frac{\partial}{\partial x_j} a_k\right) \underline{e}_i} \qquad (6.50)$$

$$\left.\begin{aligned}
[\underline{\nabla} \times \underline{a}]_x &= \frac{\partial}{\partial y} a_z - \frac{\partial}{\partial z} a_y \\
[\underline{\nabla} \times \underline{a}]_y &= \frac{\partial}{\partial z} a_x - \frac{\partial}{\partial x} a_z \\
[\underline{\nabla} \times \underline{a}]_z &= \frac{\partial}{\partial x} a_y - \frac{\partial}{\partial y} a_x
\end{aligned}\right\} \qquad (6.51)$$

ε_{yzx}

· **Deutung**: rot $\underline{a}(\underline{r}) = \underline{\nabla} \times \underline{a}(\underline{r})$ identifiziert lokal Wirbel von Vektorfeldern (6.52)

Bsp 1: $\underline{v} = \underline{\omega} \times \underline{r}$ (6.53) ... Prototyp eines Wirbels im Geschw.-feld



$\underline{\omega} = \frac{1}{2}\,\mathrm{rot}\,\underline{v}$ ... Wirbelstärke

Beweis: $\mathrm{rot}\,\underline{v} = \underline{\nabla} \times (\underline{\omega} \times \underline{r})$

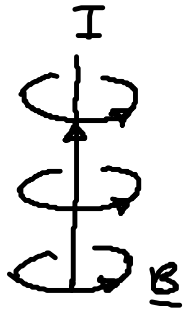
Hilfsformel: $\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}\,(\underline{a}\cdot\underline{c}) - \underline{c}\,(\underline{a}\cdot\underline{b})$

& $\underline{\nabla}$ $\underline{r}$

$$= \Big[\underline{\omega}\,\underbrace{(\underline{\nabla}\cdot\underline{r})}_{3} - \underbrace{(\underline{\omega}\cdot\underline{\nabla})\,\underline{r}}_{\omega_i \frac{\partial}{\partial x_i}\underline{r} = \underline{\omega}}\Big]$$

$$= 3\underline{\omega} - \underline{\omega} \quad = 2\underline{\omega} \quad \text{qed}$$

Bsp 2: Stromleiter:

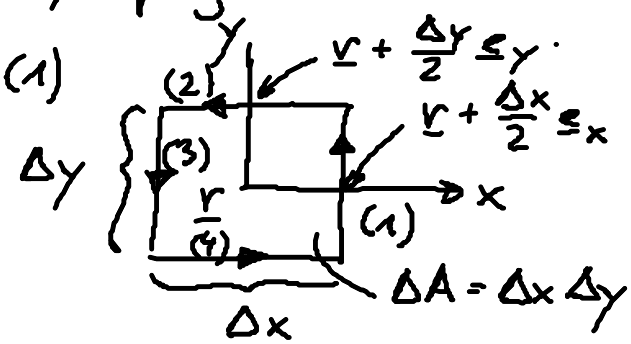


$\mathrm{rot}\,\underline{B} \sim \underline{I}$ (6.55)

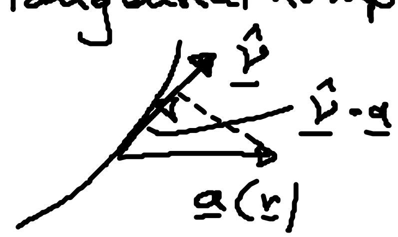
(Maxwell-Gl.)

Bsp 3: Maß für Wirbel am Flächenelement $\Delta A = \Delta x \Delta y$

(1)



(2) Tangentialkomp.



$$\boxed{\text{Summiere Tangentialkomp. von } \underline{a}(\underline{r}) \text{ an geschlossenem Weg} \longrightarrow \text{Verwirbelung}} \quad (6.56)$$

„Achtung: Weg wird im mathematisch pos. Sinne orientiert!"

$$\longrightarrow \Big[\underbrace{a_y\left(\underline{r}+\tfrac{\Delta x}{2}\underline{e}_x\right)}_{\underline{a}\cdot\underline{e}_y\ (1)} \underbrace{-\, a_y\left(\underline{r}-\tfrac{\Delta x}{2}\underline{e}_x\right)}_{\underline{a}(-\underline{e}_y)(3)}\Big]\Delta y$$

$$+\Big[\underbrace{a_x\left(\underline{r}-\tfrac{\Delta y}{2}\underline{e}_y\right)}_{\underline{a}\cdot\underline{e}_x\ (4)} - a_x\left(\underline{r}+\tfrac{\Delta y}{2}\underline{e}_y\right)\Big]\Delta x$$

$$\overset{\text{Taylor}}{=} \Big[a_y(\underline{r}) + \frac{\Delta x}{2}\frac{\partial}{\partial x}a_y - a_y(\underline{r}) + \frac{\Delta x}{2}\frac{\partial}{\partial x}a_y\Big]\Delta y$$

$$+\Big[a_x(\underline{r}) - \frac{\Delta y}{2}\frac{\partial}{\partial y}a_x - a_x(\underline{r}) - \frac{\Delta y}{2}\frac{\partial}{\partial y}a_x\Big]\Delta x$$

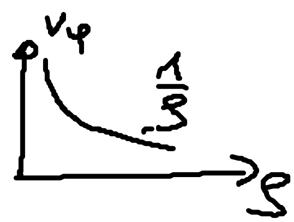
$$= \left(\frac{\partial}{\partial x}a_y - \frac{\partial}{\partial y}a_x\right)\Delta x\,\Delta y = [\mathrm{rot}\,\underline{a}]_z\,\Delta x\,\Delta y$$

$\xrightarrow[\text{von } \Delta A]{\text{bel. Orientierung}}$

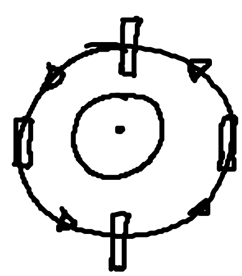
$$\boxed{\begin{array}{l} \mathrm{rot}\,\underline{a}\ \ \Delta A = \underline{\nabla}\times\underline{a}\ \Delta A \\ \text{mißt Wirbel um } \Delta A \end{array}} \qquad (6.57)$$

Bsp 4: $\underline{v}(\underline{r}) = \dfrac{g_0}{\rho}\,\underline{e}_z\times\underline{e}_\rho = \dfrac{g_0}{\rho}\,\underline{e}_\varphi$ (6.58)

o.B.: $\mathrm{rot}\,\underline{v} = 0, \quad \underline{r}\neq 0$



Deutung:



- Zylinder-/Kugelkoord.: $\mathrm{rot}\,\underline{a}$ berechenbar
- Regeln:

$$\left.\begin{array}{ll} (1) & \underline{\nabla}\times(\underline{a}+\underline{b}) = \underline{\nabla}\times\underline{a} + \underline{\nabla}\times\underline{b} \\ (2) & \underline{\nabla}\times(f(\underline{r})\,\underline{a}) = f(\underline{r})\,\underline{\nabla}\times\underline{a} + [\underline{\nabla}f(\underline{r})]\times\underline{a} \end{array}\right\} (6.59)$$

Beweis: in kartesische Koord.