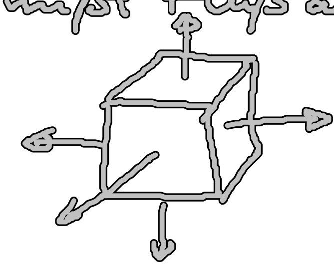
• Erinnerung:

$$\operatorname{div} \underline{a}(\underline{r}) = \underline{\nabla} \cdot \underline{a}(\underline{r})$$

kartesische Koordinaten: $\operatorname{div} \underline{a} = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z$

• Fluß = Normalkomponente von $\underline{a}$ mal Fläche (Normale immer aus Volumen heraus!)

• $\boxed{q(\underline{r}) = \operatorname{div} \underline{a} \; \Delta V \text{ mißt Fluß aus } \Delta V \text{ heraus}}$ (6.43)



$q(\underline{r}) = 0$ ... "was rein fließt, fließt raus"

> 0 ... "es fließt mehr raus als rein" $\hat{=}$ Quelle

< 0 ... Senke

Bsp: $\underline{v}(\underline{r}) = v_0 \underline{r} \longrightarrow \operatorname{div} \underline{v} = 3 v_0$
$\longrightarrow$ überall Quellen! } (6.44)

• Regeln:

$$\left.\begin{aligned} &(1)\; \underline{\nabla} \cdot (\underline{a} + \underline{b}) = \underline{\nabla} \cdot \underline{a} + \underline{\nabla} \cdot \underline{b} \\ &(2)\; \underline{\nabla} \cdot (f(\underline{r}) \underline{a}) = f(\underline{r}) \underline{\nabla} \cdot \underline{a} + \underline{a} \cdot \underline{\nabla} f(\underline{r}) \end{aligned}\right\} (6.45)$$

Beweis: in kartesischen Koord.

• Zylinderkoordinaten:

$$\underline{\nabla} \cdot \underline{a}(\underline{r}) = \left(\underline{e}_\rho \frac{\partial}{\partial \rho} + \underline{e}_\varphi \frac{1}{\rho} \frac{\partial}{\partial \varphi} + \underline{e}_z \frac{\partial}{\partial z}\right) \cdot \left(a_\rho \underline{e}_\rho + a_\varphi \underline{e}_\varphi + a_z \underline{e}_z\right)$$

<u>Achtung:</u>

$$\left.\begin{aligned} &\frac{\partial}{\partial\varphi}\underline{e}_\rho = \frac{\partial}{\partial\varphi}\begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{pmatrix} = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix} = \underline{e}_\varphi \\ &\frac{\partial}{\partial\varphi}\underline{e}_\varphi = -\underline{e}_\rho \quad \text{... nicht benötigt} \\ &\frac{\partial}{\partial x_i}\underline{e}_j = 0 \quad , \quad x_i, i = \rho, \varphi, z \end{aligned}\right\} \quad (6.46)$$

(6.46) und $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$

$$\longrightarrow \boxed{\begin{aligned} \underline{\nabla}\cdot\underline{a}(\underline{r}) &= \frac{\partial}{\partial\rho}a_\rho + \frac{1}{\rho}a_\rho + \frac{1}{\rho}\frac{\partial}{\partial\varphi}a_\varphi + \frac{\partial}{\partial z}a_z \\ &= \frac{1}{\rho}\frac{\partial(\rho a_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial}{\partial\varphi}a_\varphi + \frac{\partial}{\partial z}a_z \end{aligned}} \quad (6.47)$$

Bsp:
- $\underline{E}(\underline{r}) \sim -\frac{1}{\rho}\underline{e}_\rho \longrightarrow \operatorname{div}\underline{E}(\underline{r}) \sim \frac{1}{\rho^2} - \frac{1}{\rho^2} = 0, \ \rho \neq 0$
- $\underline{v}(\underline{r}) = \omega\rho\,\underline{e}_\varphi \longrightarrow \operatorname{div}\underline{v} = 0!$

- Kugelkoordinaten: $\underline{a} = a_r\underline{e}_r + a_\vartheta\underline{e}_\vartheta + a_\varphi\underline{e}_\varphi$ (6.48)

$$\boxed{\begin{aligned} \underline{\nabla}\cdot\underline{a}(\underline{r}) &= \frac{\partial a_r}{\partial r} + \frac{2}{r}a_r + \frac{1}{r}\frac{\partial a_\vartheta}{\partial\vartheta} + \frac{1}{r\tan\vartheta}a_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial a_\varphi}{\partial\varphi} \\ &= \frac{1}{r^2}\frac{\partial(r^2 a_r)}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial(\sin\vartheta\, a_\vartheta)}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial a_\varphi}{\partial\varphi} \end{aligned}}$$

Beweis: Übungen

Bsp: $\underline{a}(\underline{r}) = \underline{r} = r\underline{e}_r \longrightarrow \operatorname{div}\underline{a} = \frac{\partial r}{\partial r} + \frac{2}{r}r = 3!$
(vgl. 6.35)

6.6 <u>Rotation</u>

· **Def:**

$$\boxed{\begin{array}{l}\text{Rotation eines Vektorfeldes } \underline{a}(\underline{r}) \\ \quad \mathrm{rot}\,\underline{a}(\underline{r}) = \underline{\nabla} \times \underline{a}(\underline{r}) \\ \ldots \text{Vektorfeld} = \text{Wirbelfeld von } \underline{a}(\underline{r})\end{array}} \quad (6.49)$$

· Kartesische Koord.:

$$\underline{\nabla} = \underline{e}_i \frac{\partial}{\partial x_i}, \quad \underline{a}(\underline{r}) = a_i \underline{e}_i, \quad i, x_i = x, y, z$$

$$\longrightarrow \quad \underline{\nabla} \times \underline{a}(\underline{r}) = \left(\underline{e}_i \frac{\partial}{\partial x_i}\right) \times \left(a_j(\underline{r})\, \underline{e}_j\right)$$

$$\text{mit } \underline{e}_i \times \underline{e}_j = \varepsilon_{ijk}\, \underline{e}_k \quad (2.23)$$

$$\left[\frac{\partial}{\partial x_i} \underline{e}_j = 0\right] \quad = \frac{\partial}{\partial x_i} a_j(\underline{r}) \underbrace{\varepsilon_{ijk}}_{\varepsilon_{kij}} \underline{e}_k$$

ε_{312}

ε_{123}

ε_{213}

$\varepsilon_{312} = 1$

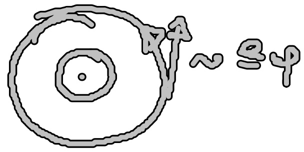
$$\xrightarrow[k \to i]{i \to j,\ j \to k} \quad \boxed{\begin{array}{l}\underline{\nabla} \times \underline{a}(\underline{r}) = \varepsilon_{ijk}\left(\frac{\partial}{\partial x_j} a_k\right) \underline{e}_i \qquad (6.50) \\ [\underline{\nabla} \times \underline{a}]_x = \frac{\partial}{\partial y} a_z - \frac{\partial}{\partial z} a_y \\ [\underline{\nabla} \times \underline{a}]_y = \frac{\partial}{\partial z} a_x - \frac{\partial}{\partial x} a_z \\ [\underline{\nabla} \times \underline{a}]_z = \frac{\partial}{\partial x} a_y - \frac{\partial}{\partial y} a_x\end{array}} \quad (6.51)$$

ε_{yzx}

· **Deutung:**

$$\boxed{\begin{array}{l}\mathrm{rot}\,\underline{a}(\underline{r}) = \underline{\nabla} \times \underline{a}(\underline{r}) \text{ identifiziert lokal} \\ \text{Wirbel von Vektorfeldern}\end{array}} \quad (6.52)$$

Bsp 1: $\underline{v} = \underline{\omega} \times \underline{r}$ (6.53) ... Prototyp eines Wirbels im Geschw.-feld



$\underline{\omega} = \frac{1}{2}\,\mathrm{rot}\,\underline{v}$ ... Wirbelstärke

Beweis: $\mathrm{rot}\,\underline{v} = \underline{\nabla} \times (\underline{\omega} \times \underline{r})$

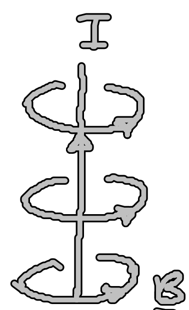
Hilfsformel: $\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}\,(\underline{a}\cdot\underline{c}) - \underline{c}\,(\underline{a}\cdot\underline{b})$

& $\underline{\nabla}$ $\underline{r}$

$$= \left[\underline{\omega}\,\underbrace{(\underline{\nabla}\cdot\underline{r})}_{3} - \underbrace{(\underline{\omega}\cdot\underline{\nabla})\,\underline{r}}_{\omega_i \frac{\partial}{\partial x_i}\underline{r} = \underline{\omega}}\right]$$

$$= 3\underline{\omega} - \underline{\omega} = 2\underline{\omega} \quad \text{qed}$$

Bsp 2: Stromleiter:

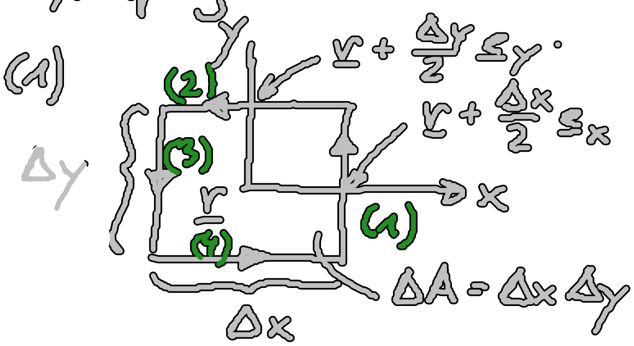


$\mathrm{rot}\,\underline{B} \sim \underline{I}$ (6.55)

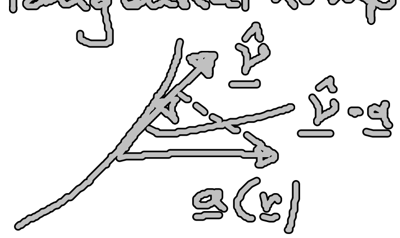
(Maxwell-Gl.)

Bsp 3: Maß für Wirbel am Flächenelement $\Delta A = \Delta x \Delta y$

(1)



(2) Tangentialkomp.



$$\boxed{\text{Summiere Tangentialkomp. von } \underline{a}(\underline{r}) \text{ am geschlossenen Weg} \longrightarrow \text{Verwirbelung}} \quad (6.56)$$

„Achtung: Weg wird im mathematisch pos. Sinne orientiert!"

$$\longrightarrow \Big[\underbrace{a_y(\underline{r}+\tfrac{\Delta x}{2}\underline{e}_x)}_{\underline{a}\cdot\underline{e}_y\ (1)} - \underbrace{a_y(\underline{r}-\tfrac{\Delta x}{2}\underline{e}_x)}_{\underline{a}(-\underline{e}_y)\ (3)}\Big]\Delta y$$

$$+\Big[\underbrace{a_x(\underline{r}-\tfrac{\Delta y}{2}\underline{e}_y)}_{\underline{a}\cdot\underline{e}_x\ (4)} - a_x(\underline{r}+\tfrac{\Delta y}{2}\underline{e}_y)\Big]\Delta x$$

$$\overset{\text{Taylor}}{=} \Big[a_y(\underline{r}) + \frac{\Delta x}{2}\frac{\partial}{\partial x}a_y - a_y(\underline{r}) + \frac{\Delta x}{2}\frac{\partial}{\partial x}a_y\Big]\Delta y$$

$$+\Big[a_x(\underline{r}) - \frac{\Delta y}{2}\frac{\partial}{\partial y}a_x - a_x(\underline{r}) - \frac{\Delta y}{2}\frac{\partial}{\partial y}a_x\Big]\Delta x$$

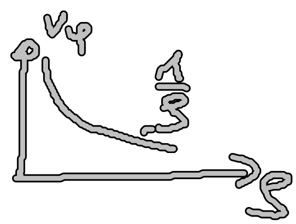
$$= \left(\frac{\partial}{\partial x}a_y - \frac{\partial}{\partial y}a_x\right)\Delta x\,\Delta y = [\operatorname{rot}\underline{a}]_z\,\Delta x\,\Delta y$$

$\xrightarrow[\text{von } \Delta A]{\text{bel. Orientierung}}$

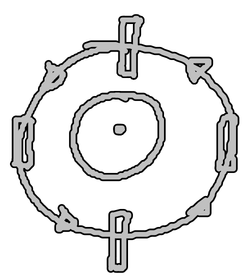
$$\boxed{\begin{array}{l}\operatorname{rot}\underline{a}\ \ \Delta A = \underline{\nabla}\times\underline{a}\ \ \Delta A \\ \text{mißt Wirbel um } \Delta A\end{array}} \quad (6.57)$$

Bsp 4: $\underline{v}(\underline{r}) = \frac{g_0}{\rho}\,\underline{e}_z\times\underline{e}_\rho = \frac{g_0}{\rho}\,\underline{e}_\varphi$ (6.58)

o. B.: $\operatorname{rot}\underline{v} = 0,\ \ \underline{r}\neq 0$



Deutung:



- Zylinder-/Kugelkoord.: $\operatorname{rot}\underline{a}$ berechenbar
- Regeln:

$$\left.\begin{array}{ll}(1) & \underline{\nabla}\times(\underline{a}+\underline{b}) = \underline{\nabla}\times\underline{a} + \underline{\nabla}\times\underline{b} \\ (2) & \underline{\nabla}\times(f(\underline{r})\,\underline{a}) = f(\underline{r})\,\underline{\nabla}\times\underline{a} + [\underline{\nabla}f(\underline{r})]\times\underline{a}\end{array}\right\}(6.59)$$

Beweis: in kartesische Koord.