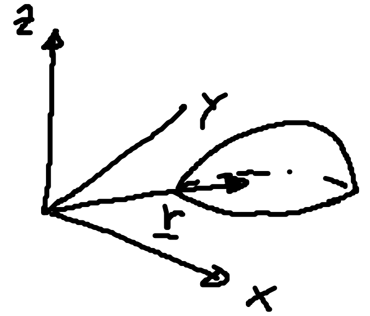
7.2. Flächenintegrale

$$\boxed{Q = \int_F \underline{a}(\underline{r}) \cdot d\underline{f} \quad \text{mit} \quad d\underline{f} = df\,\hat{\underline{\nu}},\ |\hat{\underline{\nu}}| = 1} \qquad (7.11)$$

- Fläche im Raum: $\underline{r} = \underline{r}(u,v)$ (7.12)

Bsp:

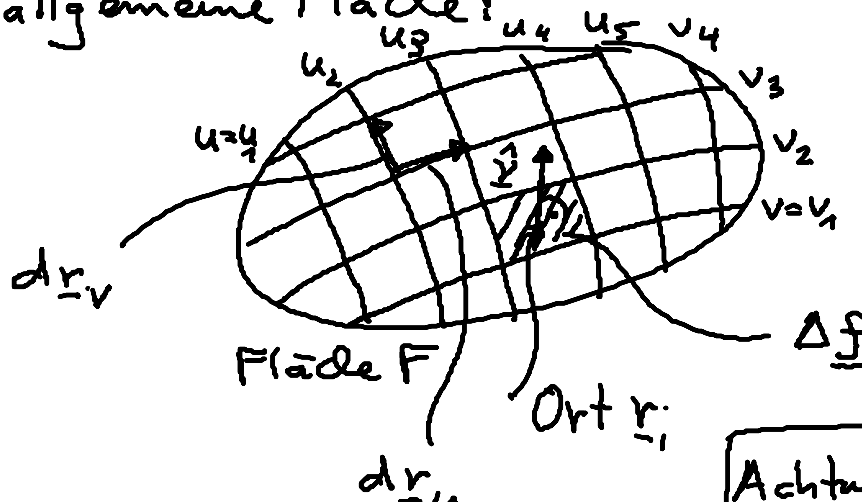


$(u,v) = (x,y)$

Bsp: $z = \sqrt{1-x^2-y^2}$

... Halbkugel

allgemeine Fläche:



$u = \text{const}$ oder $v = \text{const}$.

$\to$ Netz von Linien auf F

$\Delta\underline{f}(\underline{r}_i) = \Delta f\,\hat{\underline{\nu}},\ |\hat{\underline{\nu}}| = 1$

$$\boxed{\text{Achtung: alle } \hat{\nu} \text{ auf F in denselben Halbraum}} \qquad (7.13)$$

- Fluß von $\underline{a}(\underline{r})$ durch F:

$$Q = \sum_{\text{"}i \in F\text{"}} \underline{a}(\underline{r}_i) \cdot \Delta\underline{f}(\underline{r}_i) \qquad (7.14)$$

$$\xrightarrow{\Delta\underline{f} \to d\underline{f}} \int \underline{a}(\underline{r}) \cdot d\underline{f}(\underline{r})$$

- Berechnung:

Flächenelement $d\underline{f}$:

$d\underline{f}$:  $d\underline{f}$

$d\underline{r}_v = \frac{\partial \underline{r}}{\partial v}\, dv \qquad d\underline{r}_u = \frac{\partial \underline{r}}{\partial u}\, du$

$\frac{\partial \underline{r}}{\partial u} \approx \frac{\underline{r}(u+du) - \underline{r}(u)}{du}$

$$\longrightarrow \boxed{\begin{aligned} d\underline{f} &= d\underline{r}_u \times d\underline{r}_v \\ &= \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right) du\, dv \end{aligned}} \quad (7.16)$$

$$\xrightarrow{\text{in (7.11)}} \boxed{Q = \int_F \underline{a}(\underline{r}) \cdot \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right) du\, dv} \quad (7.17)$$

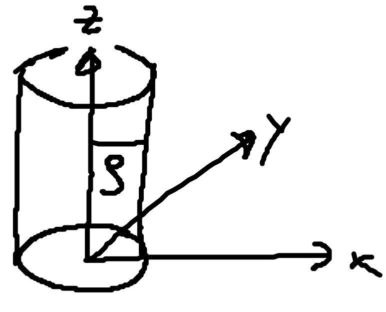
... Doppelintegral (s. HM)

· Bsp: für $d\underline{f}$:

(1) Fläche ∥ xy-Ebene: $(u,v) = (x,y) \longrightarrow d\underline{f} = \underline{e}_z\, dx\, dy$

(2) Zylinderoberfläche um $\underline{e}_z$ mit Radius ϱ:

$(u,v) = (\varphi, z)$



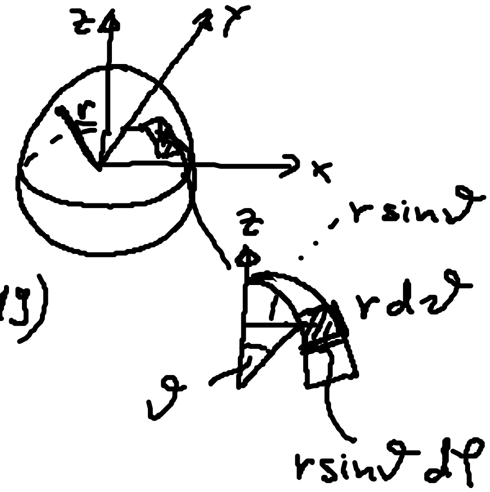
$$\underline{r} = \varrho\, \underline{e}_\varrho + z\, \underline{e}_z = \begin{pmatrix} \varrho\cos\varphi \\ \varrho\sin\varphi \\ z \end{pmatrix}$$

$$\left.\begin{aligned} \frac{\partial \underline{r}}{\partial \varphi} &= \varrho \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \varrho\, \underline{e}_\varphi \\ \frac{\partial \underline{r}}{\partial z} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underline{e}_z \end{aligned}\right\} \quad \begin{aligned} d\underline{f} &= \varrho\, \underbrace{\underline{e}_\varphi \times \underline{e}_z}_{\underline{e}_\varrho}\, d\varphi\, dz \\ &\boxed{d\underline{f} = \underline{e}_\varrho\, \underbrace{\varrho\, d\varphi\, dz}} \quad (7.18) \end{aligned}$$

df auf Zylinderoberfläche

(3) Kugeloberfläche um $\underline{r} = 0$ mit Radius r:

$(u,v) = (\vartheta, \varphi)$

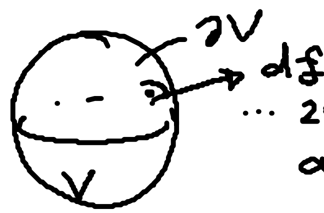


$$\xrightarrow{\text{o.B.}} \boxed{\begin{aligned} d\underline{f} &= \underline{e}_r \, r^2 \sin\vartheta \, d\vartheta \, d\varphi \\ &= -\underline{e}_r \, r^2 \, d\cos\vartheta \, d\varphi \end{aligned}} \quad (7.19)$$

$d\cos\vartheta = -\sin\vartheta \, d\vartheta$

- geschlossene Fläche ∂V um Volumen V

$$\boxed{\int_{\partial V} \underline{a}(\underline{r}) \cdot d\underline{f} \qquad (7.20)}$$



… zeigt immer aus V hinaus! ☺

- Bsp: Oberfläche einer Kugel mit Radius r:
  nimm $\underline{a}(\underline{r}) = \underline{e}_r$!

$$\longrightarrow F_K = \int_{\partial V_K} \underline{e}_r \cdot d\underline{f} \, \underline{e}_r = \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi} r^2 \sin\vartheta \, d\vartheta \, d\varphi$$

$$\underset{=}{(\text{hier})} \; r^2 \int_{\varphi=0}^{2\pi} d\varphi \int_{\vartheta=0}^{\pi} \sin\vartheta \, d\vartheta = 2\pi r^2 \, (-1) \int_1^{-1} d\cos\vartheta$$

$$= 2\pi r^2 \int_{-1}^{1} d\cos\vartheta = 2\pi r^2 \cos\vartheta \Big|_{-1}^{1} = 4\pi r^2!$$

7.3 Satz von Stokes

- Satz:

Fluß von rot $\underline{a}$ durch F

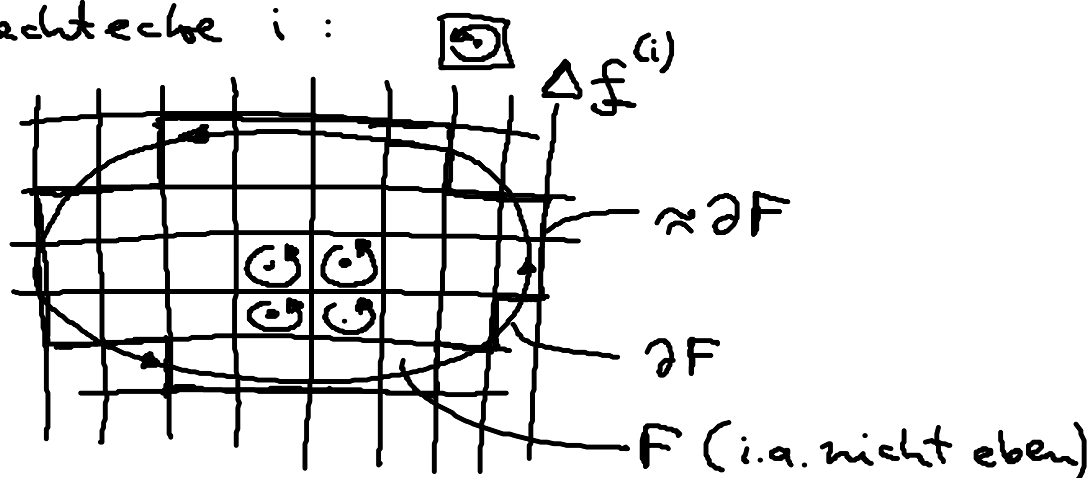
$$\int_F \text{rot}\,\underline{a}\cdot d\underline{f} = \oint_{C=\partial F} \underline{a}\cdot d\underline{r} \qquad (7.21)$$

... Zirkulation von $\underline{a}$ entlang Randkurve ∂F von F

wichtig: (1) rot $\underline{a}$ definiert auf ganz F
(2) Umlaufsinn von $C=\partial F$ über Rechte-Hand-Regel für $d\underline{f}$ von F

- Beweis:

1. Nähere F durch "viele" gleich orientierte Rechtecke i:

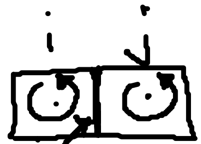


2. $$\int_F \text{rot}\,\underline{a}\cdot d\underline{f} \approx \sum_i \text{rot}\,\underline{a}(\underline{r}_i)\cdot\Delta\underline{f}^{(i)}$$
(über Rechtecke)

3. $$\text{rot}\,\underline{a}(\underline{r}_i)\cdot\Delta\underline{f}^{(i)} = \text{"}\underbrace{\oint_{(i)} \underline{a}\cdot d\underline{r}^{(i)}}_{\text{Zirkulation um Rechteck } i}\text{"}$$

(vgl. Bsp. 3, Kap 6.6)

4. benachbarte Flächenelemente: i & j



$$d\underline{r}^{(i)} = -d\underline{r}^{(j)} \longrightarrow \underline{a}\cdot d\underline{r}^{(i)} + \underline{a}\cdot \underbrace{d\underline{r}^{(j)}}_{-d\underline{r}^{(i)}} = 0$$

also: in $\sum_i \mathrm{rot}\,\underline{a}\cdot\Delta f^{(i)} = \sum_i \text{"}\oint_{(i)} \underline{a}\cdot d\underline{r}^{(i)}\text{"}$

nur "freiliegende" Kurventeile der (i) tragen bei

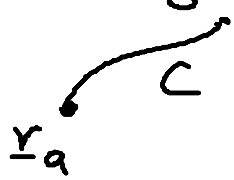
$\longrightarrow$ Rand $C = \partial F$

$$\longrightarrow \sum_i \text{"}\oint \underline{a}\cdot d\underline{r}^{(i)}\text{"} \longrightarrow \oint_{\partial F} \underline{a}\cdot d\underline{r} \quad \text{qed}$$

· Anwendungen:

(1) $\underline{F} = -\mathrm{grad}\,U$ ... Kraftfeld

a) beliebiger Weg: $\int_C \underline{F}\cdot d\underline{r} = -\int_C \underbrace{\mathrm{grad}\,U\cdot d\underline{r}}_{=dU}$ (6.20



$$= -\int_C dU = -U(\underline{r})\Big|_{U(\underline{r}_a)}^{U(\underline{r}_e)} =$$

$$= -\left[U(\underline{r}_e) - U(\underline{r}_a)\right]$$

b) geschlossener Weg: $\underline{r}_a = \underline{r}_e$

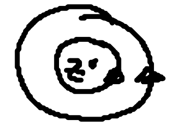
$$\longrightarrow \oint_{C=\partial F} \underline{F}\cdot d\underline{r} = 0 \qquad (7.22)$$

c) $\mathrm{rot}\,\underline{F} = -\mathrm{rot}\,\mathrm{grad}\,U = 0$ vgl. (6.60)

$$\longrightarrow \int_F \mathrm{rot}\,\underline{F}\cdot d\underline{f} = 0$$

$\xrightarrow{\text{mit (7.22)}}$ Stokes o.k. ✓

(2) Achtung: $\underline{v}(\underline{r}) = \frac{g_0}{\rho}\,\underline{e}_\varphi$ vgl. (6.58)

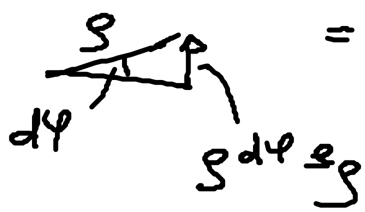


a) $\mathrm{rot}\,\underline{v} = 0, \quad \underline{r} \neq z\,\underline{e}_z$

$\longrightarrow \int_F \mathrm{rot}\,\underline{v} \cdot d\underline{f}$ nicht berechenbar falls $\underline{r} = z\,\underline{e}_z \in F$

b) $C = \partial F$: Kreis um z-Achse mit $\rho = \mathrm{const}$

$$\int_{C=\partial F} \underline{v}(\underline{r}) \cdot \underbrace{d\underline{r}'}_{\underline{e}_\varphi \rho\, d\varphi} = \int \frac{g_0}{\cancel{\rho}} \underbrace{\underline{e}_\varphi \cdot \underline{e}_\varphi}_{=1} \cancel{\rho}\, d\varphi$$



$$= g_0 \int_0^{2\pi} d\varphi = 2\pi g_0 \neq 0$$