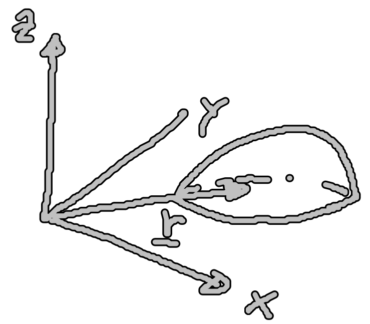
7.2. Flächenintegrale

- $$\boxed{Q = \int_F \underline{a}(\underline{r}) \cdot d\underline{f} \quad \text{mit} \quad d\underline{f} = df\, \hat{\underline{\nu}},\ |\hat{\underline{\nu}}| = 1} \qquad (7.11)$$

- Fläche im Raum: $\underline{r} = \underline{r}(u,v)$ (7.12)

Bsp:



$(u,v) = (x,y)$

Bsp: $z = \sqrt{1 - x^2 - y^2}$

... Halbkugel

allgemeine Fläche:

$u = \text{const}$ oder $v = \text{const}$.

$\rightarrow$ Netz von Linien auf F

$\Delta \underline{f}(\underline{r}_i) = \Delta f\, \hat{\underline{\nu}},\ |\hat{\underline{\nu}}| = 1$

$\boxed{\text{Achtung: alle } \hat{\nu} \text{ auf F in denselben Halbraum}}$ (7.13)

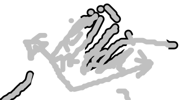
- Fluß von $\underline{a}(\underline{r})$ durch F:

$$Q = \sum_{\text{"}i \in F\text{"}} \underline{a}(\underline{r}_i) \cdot \Delta \underline{f}(\underline{r}_i)$$

$$\xrightarrow{\Delta \underline{f} \to d\underline{f}} \int \underline{a}(\underline{r}) \cdot d\underline{f}(\underline{r}) \qquad (7.14)$$

- Berechnung:

Flächenelement $d\underline{f}$:

$d\underline{f}$:  $d\underline{f}$

$d\underline{r}_v = \frac{\partial \underline{r}}{\partial v}\, dv$ $\qquad$ $d\underline{r}_u = \frac{\partial \underline{r}}{\partial u}\, du$

$$\frac{\partial \underline{r}}{\partial u} \approx \frac{\underline{r}(u+du) - \underline{r}(u)}{du}$$

$$\longrightarrow \quad \boxed{\begin{aligned} d\underline{f} &= d\underline{r}_u \times d\underline{r}_v \\ &= \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right) du\, dv \end{aligned}} \qquad (7.16)$$

$$\xrightarrow{\text{in (7.11)}} \quad \boxed{Q = \int_F \underline{a}(\underline{r}) \cdot \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right) du\, dv} \qquad (7.17)$$

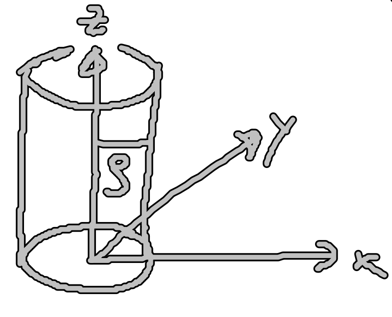
... Doppelintegral (s. HM)

- Bsp: für $d\underline{f}$:

(1) Fläche ∥ xy-Ebene: $(u,v) = (x,y) \longrightarrow d\underline{f} = \underline{e}_z\, dx\, dy$

(2) Zylinderoberfläche um $\underline{e}_z$ mit Radius ϱ:

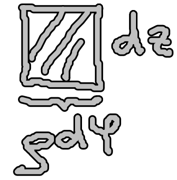
$(u,v) = (\varphi, z)$



$$\underline{r} = \varrho\, \underline{e}_\varrho + z\, \underline{e}_z = \begin{pmatrix} \varrho \cos\varphi \\ \varrho \sin\varphi \\ z \end{pmatrix}$$

$$\left.\begin{aligned} \frac{\partial \underline{r}}{\partial \varphi} &= \varrho \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \varrho\, \underline{e}_\varphi \\ \frac{\partial \underline{r}}{\partial z} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underline{e}_z \end{aligned}\right\} \quad \begin{aligned} d\underline{f} &= \varrho\, \underbrace{\underline{e}_\varphi \times \underline{e}_z}_{\underline{e}_\varrho}\, d\varphi\, dz \\ &\boxed{d\underline{f} = \underline{e}_\varrho\, \underbrace{\varrho\, d\varphi\, dz}} \end{aligned} \qquad (7.18)$$

df auf Zylinderoberfläche



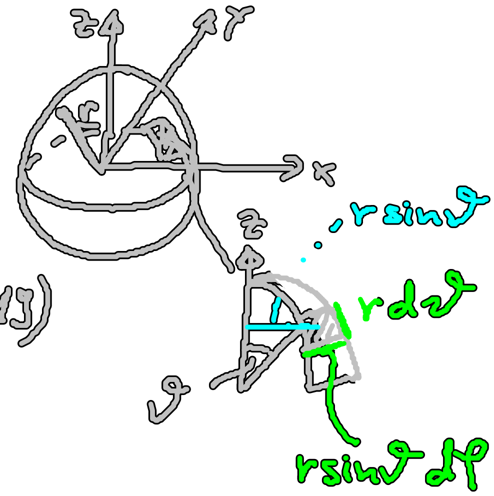
(3) Kugeloberfläche um $\underline{r} = 0$ mit Radius r:

$(u,v) = (\vartheta, \varphi)$

$\xrightarrow{\text{o.B.}}$

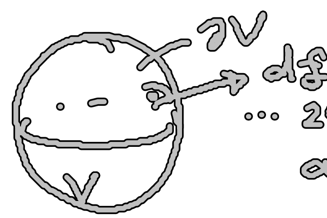
$$\boxed{\begin{aligned} d\underline{f} &= \underline{e}_r \; r^2 \sin\vartheta \, d\vartheta \, d\varphi \\ &= -\underline{e}_r \; r^2 \, d\cos\vartheta \, d\varphi \end{aligned}} \quad (7.19)$$

$d\cos\vartheta = -\sin\vartheta \, d\vartheta$



- geschlossene Fläche ∂V um Volumen V

$$\boxed{\int_{\partial V} \underline{a}(\underline{r}) \cdot d\underline{f} \qquad (7.20)}$$



... zeigt immer aus V hinaus!

- Bsp: Oberfläche einer Kugel mit Radius r:
  nimm $\underline{a}(\underline{r}) = \underline{e}_r$!

$$\longrightarrow F_K = \int_{\partial V_K} \underline{e}_r \cdot d\underline{f}\, \underline{e}_r = \int_{\varphi=0}^{2\pi}\int_{\vartheta=0}^{\pi} r^2 \sin\vartheta \, d\vartheta \, d\varphi$$

$$\underset{\text{(hier)}}{=} r^2 \int_{\varphi=0}^{2\pi} d\varphi \int_{\vartheta=0}^{\pi} \sin\vartheta \, d\vartheta = 2\pi r^2 (-1) \int_{1}^{-1} d\cos\vartheta$$

$$= 2\pi r^2 \int_{-1}^{1} d\cos\vartheta = 2\pi r^2 \cos\vartheta \Big|_{-1}^{1} = 4\pi r^2.$$

7.3 Satz von Stokes

- **Satz:** Fluß von rot $\underline{g}$ durch F

$$\int_F \text{rot}\,\underline{g} \cdot d\underline{f} = \oint_{C=\partial F} \underline{g} \cdot d\underline{r} \qquad (7.21)$$

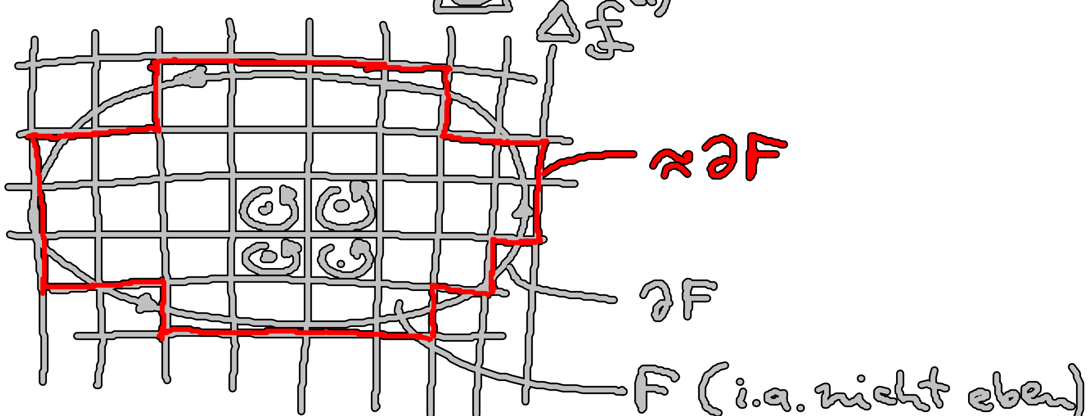
... Zirkulation von $\underline{g}$ entlang Randkurve ∂F von F

wichtig: (1) rot $\underline{g}$ definiert auf ganz F

(2) Umlaufsinn von $C=\partial F$ über Rechte-Hand-Regel für $d\underline{f}$ von F

- Beweis:

1. Nähere F durch "viele" gleich orientierte Rechtecke i: $\Delta\underline{f}^{(i)}$

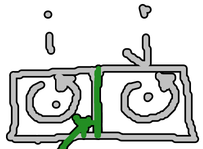


2. $$\int_F \text{rot}\,\underline{g} \cdot d\underline{f} \approx \sum_{i} \text{rot}\,\underline{g}(\underline{r}_i) \cdot \Delta\underline{f}^{(i)}$$
(über Rechtecke)

3. $$\text{rot}\,\underline{g}(\underline{r}_i) \cdot \Delta\underline{f}^{(i)} = \text{"}\underbrace{\oint_{(i)} \underline{g} \cdot d\underline{r}^{(i)}}_{\text{Zirkulation um Rechteck } i}\text{"}$$

(vgl. Bsp. 3, Kap 6.6)

4. benachbarte Flächenelemente: i & j



$d\underline{r}^{(j)} = -d\underline{r}^{(i)} \longrightarrow \quad \underline{a}\cdot d\underline{r}^{(i)} + \underline{a}\cdot \underbrace{d\underline{r}^{(j)}}_{-d\underline{r}^{(i)}} = 0$

also: in $\sum_i \operatorname{rot}\underline{a}\cdot\Delta\underline{f}^{(i)} = \sum_i \text{"}\oint_{(i)} \underline{a}\cdot d\underline{r}^{(i)}\text{"}$

nur "freiliegende" Kurventeile der (i) tragen bei

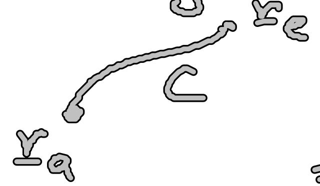
$\longrightarrow$ Rand $C = \partial F$

$\longrightarrow \quad \sum_i \text{"}\oint \underline{a}\cdot d\underline{r}^{(i)}\text{"} \longrightarrow \oint_{\partial F} \underline{a}\cdot d\underline{r}$ qed

- Anwendungen:

(1) $\underline{F} = -\operatorname{grad} U$ ... Kraftfeld

a) beliebiger Weg: $\int_C \underline{F}\cdot d\underline{r} = -\int_C \underbrace{\operatorname{grad} U\cdot d\underline{r}}_{= dU \ (6.20)}$



$$= -\int_C dU = -U(\underline{r})\Big|_{U(\underline{r}_a)}^{U(\underline{r}_e)} =$$

$$= -\left[U(\underline{r}_e) - U(\underline{r}_a)\right]$$

b) geschlossener Weg: $\underline{r}_a = \underline{r}_e$

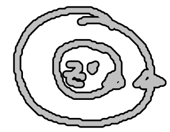
$$\longrightarrow \oint_{C=\partial F} \underline{F}\cdot d\underline{r} = 0 \qquad (7.22)$$

c) $\operatorname{rot}\underline{F} = -\operatorname{rot}\operatorname{grad} U = 0$ vgl. (6.60)

$$\longrightarrow \int_F \operatorname{rot}\underline{F}\cdot d\underline{f} = 0$$

mit (7.22) $\longrightarrow$ Stokes o.k. ✓

(2) Achtung: $\underline{v}(\underline{r}) = \frac{g_0}{\varrho} \underline{e}_\varphi$ vgl. (6.58)

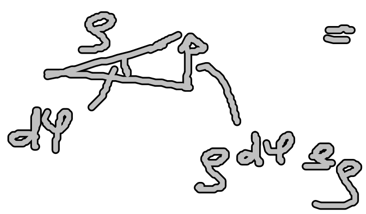


a) $\operatorname{rot} \underline{v} = 0, \quad \underline{r} \neq z \underline{e}_z$

$$\longrightarrow \int_F \operatorname{rot} \underline{v} \cdot d\underline{f}$$ nicht berechenbar falls $\underline{r} = z \underline{e}_z \in F$

b) $C = \partial F$: Kreis um z-Achse mit $\varrho = \text{const}$

$$\int_{C=\partial F} \underline{v}(\underline{r}) \cdot \underbrace{d\underline{r}'}_{\underline{e}_\varphi \varrho \, d\varphi} = \int \frac{g_0}{\cancel{\varrho}} \underbrace{\underline{e}_\varphi \cdot \underline{e}_\varphi}_{=1} \cancel{\varrho} \, d\varphi$$



$$= g_0 \int_0^{2\pi} d\varphi = 2\pi g_0 \neq 0$$