Poisson-Gleichung $\quad \Delta\phi(\underline{r}) = -\frac{1}{\epsilon_0}\rho(\underline{r})$

$$G(\underline{r}-\underline{r}') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\underline{r}-\underline{r}'|}$$

<u>Beweis</u>:

$$\Delta\phi(\underline{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathbb{R}^3} d^3r'\,\rho(\underline{r}')\,\Delta_r\frac{1}{|\underline{r}-\underline{r}'|}$$

Problem: Singularität bei $\underline{r}=\underline{r}'$!

a) $\underline{r}\neq\underline{r}'$:

$$-\Delta_r\frac{1}{|\underline{r}-\underline{r}'|} = -\underline{\nabla}_r\cdot\underbrace{\underline{\nabla}_r\frac{1}{|\underline{r}-\underline{r}'|}} = \underline{\nabla}_r\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3}$$

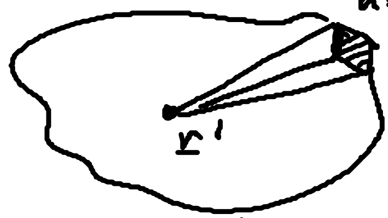
$$= \frac{\underline{\nabla}_r\cdot(\underline{r}-\underline{r}')}{|\underline{r}-\underline{r}'|^3} + (\underline{r}-\underline{r}')\,\underline{\nabla}_r\frac{1}{|\underline{r}-\underline{r}'|^3}$$

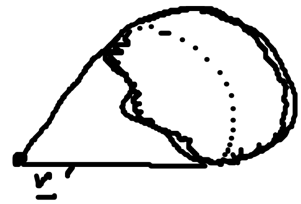
$$= \frac{3}{|\underline{r}-\underline{r}'|^3} - 3(\underline{r}-\underline{r}')\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^5} = 0$$

b) $$\int_V d^3r\,\Delta_r\frac{1}{|\underline{r}-\underline{r}'|} = \underbrace{\int_V d^3r\,\underline{\nabla}\cdot\underline{\nabla}\frac{1}{|\underline{r}-\underline{r}'|}} \overset{\text{Gauß}}{=} \underbrace{\oint_{\partial V} d\underline{f}\cdot\underline{\nabla}\frac{1}{|\underline{r}-\underline{r}'|}}$$

$$= -\oint_{\partial V} d\underline{f}\cdot \underbrace{\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3}} = -\oint d\Omega = \begin{cases} -4\pi & (\underline{r}'\in V) \\ =0 & (\underline{r}'\notin V)\end{cases}$$

$$d f_\perp = r^2 d\Omega$$





<u>Erg</u>: $\Delta_r \frac{1}{|\underline{r}-\underline{r}'|} = -4\pi\,\delta(\underline{r}-\underline{r}')$

Dirac'sche δ-Fkt. (δ-Distribution)

$$\int_V d^3r\, \delta(\underline{r}-\underline{r}_0) = \begin{cases} 1 & \underline{r}_0\in V \\ 0 & \text{sonst}\end{cases}$$

$$\int_V d^3r\, f(\underline{r})\,\delta(\underline{r}-\underline{r}_0) = f(\underline{r}_0) \quad \text{falls } \underline{r}_0\in V$$

Also

$$\Delta\phi(\underline{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathbb{R}^3} d^3r'\, \rho(\underline{r}')\, \underbrace{\Delta_r \frac{1}{|\underline{r}-\underline{r}'|}}$$

$$= \frac{-4\pi}{4\pi\epsilon_0}\int_{\mathbb{R}^3} d^3r'\, \rho(\underline{r}')\,\delta(\underline{r}-\underline{r}')$$

$$= -\frac{1}{\epsilon_0}\rho(\underline{r})$$

Poisson-Gl. erfüllt $\square$

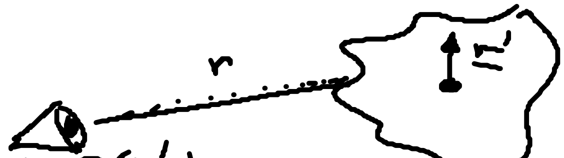
Erg.

$$\boxed{G(\underline{r}-\underline{r}') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\underline{r}-\underline{r}'|}}$$

zu den Randbed. $\phi \xrightarrow{r\to\infty} 0$

1.4. <u>Elektrische Multipol-Entwicklung</u>

räumlich begrenzte Ladungsverteilung $\rho(\underline{r}')$
in der Umgebung von $\underline{r}' = 0$



Frage: asymptot. Verhalten
von $\phi(\underline{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\rho(\underline{r}')}{|\underline{r}-\underline{r}'|}$ für $r \to \infty$

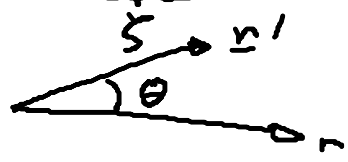
Methode: Entwickl. des Integranden in Taylorreihe
für $r \gg r'$

$$G(\underline{r}-\underline{r}') = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \left(\underline{r}' \cdot \underline{\nabla}_r\right)^l G(\underline{r})$$

$$\phi(\underline{r}) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \int d^3r' \left(\underline{r}' \cdot \underline{\nabla}_r\right)^l G(\underline{r})\,\rho(\underline{r}')$$

explizit mit $G(\underline{r}-\underline{r}') = \frac{1}{4\pi\epsilon_0 |\underline{r}-\underline{r}'|}$:

$$\frac{1}{|\underline{r}-\underline{r}'|} = \left(r^2 - 2\underline{r}\cdot\underline{r}' + r'^2\right)^{-1/2} = \frac{1}{r}\left(1 - 2\frac{r'}{r}\underline{\cos\theta} + \frac{r'^2}{r^2}\right)^{-1/2}$$



Konvergente Reihe für $r' < r$, $|\zeta| < 1$

$$\underbrace{\left(1 - 2\frac{r'}{r}\zeta + \left(\frac{r'}{r}\right)^2\right)^{-1/2}}_{\text{Erzeugende}} = \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l P_l(\zeta)$$

$\Longrightarrow$ Legendre-Polynome $P_l(\zeta)$
(Kugelfunktionen)

Es gilt: $$P_l(\zeta) = \frac{1}{l!}\left[\frac{\partial^l}{\partial t^l}\left(1 - 2t\zeta + t^2\right)^{-1/2}\right]_{t=0}$$

insbesondere
$$P_0(\zeta) = 1$$
$$P_1(\zeta) = \zeta = \cos\theta$$
$$P_2(\zeta) = \frac{1}{2}\left(3\zeta^2 - 1\right) = \frac{1}{4}\left(3\cos 2\theta + 1\right)$$

also

$$\phi(\underline{r}) = \frac{1}{4\pi\epsilon_0}\,\frac{1}{r}\int d^3r'\,\rho(\underline{r}')\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^{l} P_l(\cos\theta)$$

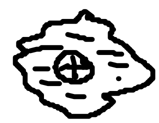
$$= \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty} Q_l\, r^{-l-1}$$

mit $Q_l = \int d^3r'\; r'^{\,l}\,\rho(\underline{r}')\, P_l(\cos\theta)$ $\quad 2^l$-Pole

Entwicklung nach Potenzen von r !

<u>$l = 0$</u> : $\phi^{(0)}(\underline{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q_0}{r}$, $\quad Q_0 = \int d^3r'\,\rho(\underline{r}')$ (Gesamtladung)

fällt am langsamsten ab! <u>Monopol</u>

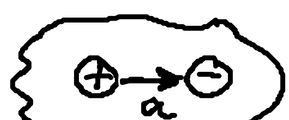


<u>$l = 1$</u> : $\phi^{(1)}(\underline{r}) = \frac{1}{4\pi\epsilon_0}\frac{\underline{p}\cdot\underline{r}}{r^3}$

$$Q_1 = \int d^3r'\,\rho(\underline{r}')\underbrace{r'\cos\theta}_{\frac{\underline{r}'\cdot\underline{r}}{r}} = \frac{\underline{p}\cdot\underline{r}}{r}$$

$$\underline{p} := \int d^3r'\,\rho(\underline{r}')\,\underline{r}' \qquad \underline{\text{Dipolmoment}}$$

fällt $\sim \frac{1}{r^2}$



$(Q_0 = 0)$

wichtigster Term für insgesamt neutrale Körper

<u>Beispiel</u>: 2 Punktladung q, $-q$ bei $\underline{r}_1$, $\underline{r}_2$:

$$\rho(\underline{r}') = q\left[\delta(\underline{r}' - \underline{r}_1) - \delta(\underline{r}' - \underline{r}_2)\right]$$

$$Q_0 = 0 \,, \quad \underline{p} = q\,(\underline{r}_1 - \underline{r}_2) = q\,\underline{a}$$

<u>Feld</u> des Dipolpotentials:

$$E_i = -\frac{1}{4\pi\epsilon_0}\,\partial_i \frac{p_k x_k}{r^3} = \frac{1}{4\pi\epsilon_0}\left(\frac{3 x_i\, p_k x_k}{r^5} - \delta_{ik}\frac{p_k}{r^3}\right)$$

(Summ. konv.)

$$\underline{E}(\underline{r}) = \frac{1}{4\pi\epsilon_0}\,\frac{1}{r^5}\left(3\,(\underline{p}\cdot\underline{r})\,\underline{r} - r^2\,\underline{p}\right)$$

$$\sim \frac{1}{r^3} \qquad \text{für } r \to \infty$$

$\underline{\ell = 2}$

$$\underline{\ell = 2} : \quad \phi^{(2)}(\underline{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q_2}{r^3}$$

$$Q_2 = \frac{1}{2} \int d^3r' \rho(\underline{r}') (r')^2 (3\cos^2\theta - 1)$$

$$= \frac{1}{2} \int d^3r' \rho(\underline{r}') \Big(3 \underbrace{\frac{\underline{r}' \cdot \underline{n}}{r} \frac{\underline{r}' \cdot \underline{n}}{r}}_{\frac{x_k' x_k x_\ell' x_\ell}{r^2}} - (r')^2 \Big)$$

$$= \frac{1}{2r^2} \int d^3r' \rho(\underline{r}') (\underline{3} x_k' x_\ell' - (r')^2 \delta_{k\ell}) \overbrace{x_k x_\ell}^{2}$$

Q_{kl} Quadrupolmoment

(Spurfrei, symm. Tensor:
$\sum_{i=1}^{3} Q_{ii} = \int d^3r' \rho(\underline{r}')(3r'^2 - 3(r')^2) = 0$)

$\Downarrow$

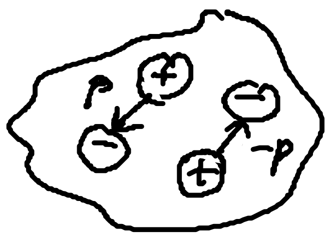
es ex. orthogonale Koord.-trafo auf Diagonalform: $Q_{kl} = 0 \quad (k \neq l)$

$\Rightarrow$ nur 2 unabh. Komp.

$$\phi^{(2)}(\underline{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{2r^5} Q_{kl} x_k x_l = \frac{1}{4\pi\epsilon_0} \frac{\underline{r}\,\underline{\underline{Q}}\,\underline{r}}{2r^5}$$

$$\sim \frac{1}{r^3}$$

<u>Beispiel</u>: 2 entgegengerichtete Dipole:



1.5 Elektrostat. Feldenergie

Kraft $\underline{F}(\underline{r}) = q\underline{E}(\underline{r}) = -q\underline{\nabla}\phi(\underline{r})$

$\Rightarrow V(\underline{r}) = q\phi(\underline{r})$ pot. Energie einer Ladung q im el. Feld $\underline{E}(\underline{r})$

$$W_{ij} = q_i \frac{1}{4\pi\epsilon_0} \frac{q_j}{|\underline{r}_i - \underline{r}_j|} = W_{ji}$$

Gesamte pot. Energie eines Systems von Ladungen

$$W = \frac{1}{2} \sum_{\substack{ij \\ i \neq j}} W_{ij} = \frac{1}{8\pi\epsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\underline{r}_i - \underline{r}_j|}$$

Kontinuierliche Ladungsverteilung $\varrho(\underline{r})$:

$$W = \frac{1}{8\pi\epsilon_0} \int d^3r \int d^3r' \frac{\varrho(\underline{r})\varrho(\underline{r}')}{|\underline{r}-\underline{r}'|}$$

$$\boxed{W = \frac{1}{2}\int d^3r\, \phi(r)\,\varrho(\underline{r})}$$

Mit $\varrho = \epsilon_0 \underline{\nabla}\cdot\underline{E}$ folgt:

$$W = \frac{\epsilon_0}{2}\int d^3r\, \phi(\underline{r})\, \underline{\nabla}\cdot E$$

$$= \frac{\epsilon_0}{2}\left[\int_{\mathbb{R}^3} d^3r\, \underline{\nabla}(\phi\underline{E}) - \int d^3r\,(\nabla\phi)\,\underline{E}(\underline{r})\right]$$

$$\overset{\text{gauß}}{=} \frac{\epsilon_0}{2}\Big[\underbrace{\int_{S_\infty} d\underline{f}\,(\overset{\sim r^{-1}}{\phi}\,\overset{\sim r^{-2}}{\underline{E}})}_{0} + \int d^3r\,(\underline{E}(\underline{r}))^2\Big]$$

$$= \int d^3r\, w(\underline{r})$$

mit $w(\underline{r}) = \frac{\epsilon_0}{2}(\underline{E}(\underline{r}))^2$ = Energiedichte des el. Feldes