Poisson-Gleichung $\Delta\phi(\underline{r}) = -\frac{1}{\epsilon_0}\rho(\underline{r})$

$$G(\underline{r}-\underline{r}') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\underline{r}-\underline{r}'|}$$

Beweis:

$$\Delta\phi(\underline{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathbb{R}^3} d^3r'\,\rho(\underline{r}')\,\Delta_r\frac{1}{|\underline{r}-\underline{r}'|}$$

Problem: Singularität bei $\underline{r}=\underline{r}'$!

a) $\underline{r}\neq\underline{r}'$:

$$-\Delta_r\frac{1}{|\underline{r}-\underline{r}'|} = -\underline{\nabla}_r\cdot\underbrace{\underline{\nabla}_r\frac{1}{|\underline{r}-\underline{r}'|}}_{} = \underline{\nabla}_r\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3}$$

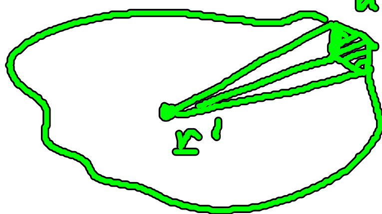
$$= \frac{\underline{\nabla}_r\cdot(\underline{r}-\underline{r}')}{|\underline{r}-\underline{r}'|^3} + (\underline{r}-\underline{r}')\,\underline{\nabla}_r\frac{1}{|\underline{r}-\underline{r}'|^3}$$

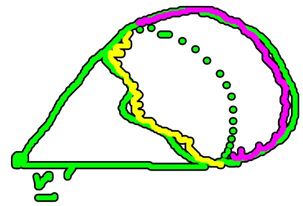
$$= \frac{3}{|\underline{r}-\underline{r}'|^3} - 3(\underline{r}-\underline{r}')\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^5} = 0$$

b) $$\int_V d^3r\,\Delta_r\frac{1}{|\underline{r}-\underline{r}'|} = \underbrace{\int_V d^3r\,\underline{\nabla}\cdot}_{}\underline{\nabla}\frac{1}{|\underline{r}-\underline{r}'|} \overset{\text{Gauß}}{=} \underbrace{\oint_{\partial V} d\underline{f}\cdot}_{}\underline{\nabla}\frac{1}{|\underline{r}-\underline{r}'|}$$

$$= -\oint_{\partial V} \underbrace{d\underline{f}\cdot\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3}} = -\oint d\Omega = \begin{cases} -4\pi & (\underline{r}'\in V)\\ =0 & (\underline{r}'\notin V)\end{cases}$$

$df_\perp = r^2 d\Omega$





Erg: $\Delta_r \frac{1}{|\underline{r}-\underline{r}'|} = -4\pi\,\delta(\underline{r}-\underline{r}')$

Dirac'sche δ-Fkt. (δ-Distribution)

$$\int_V d^3r\,\delta(\underline{r}-\underline{r}_0) = \begin{cases} 1 & \underline{r}_0\in V\\ 0 & \text{sonst}\end{cases}$$

$$\int_V d^3r\, f(\underline{r})\,\delta(\underline{r}-\underline{r}_0) = f(\underline{r}_0) \quad \text{falls } \underline{r}_0\in V$$

Also
$$\Delta\Phi(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathbb{R}^3} d^3r'\,\rho(\underline{r}')\,\underbrace{\Delta_r\frac{1}{|\underline{r}-\underline{r}'|}}$$

$$= \frac{-4\pi}{4\pi\varepsilon_0}\int_{\mathbb{R}^3} d^3r'\,\rho(\underline{r}')\,\delta(\underline{r}-\underline{r}')$$

$$= -\frac{1}{\varepsilon_0}\rho(\underline{r})$$

Poisson-Gl. erfüllt

□

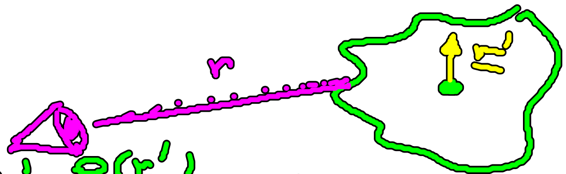
Erg.
$$\boxed{G(\underline{r}-\underline{r}') = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{|\underline{r}-\underline{r}'|}}$$

zu den Randbed. $\Phi \xrightarrow{r\to\infty} 0$

1.4. Elektrische Multipol-Entwicklung

räumlich begrenzte Ladungsverteilung $\rho(\underline{r}')$
in der Umgebung von $\underline{r}'=0$

Frage: asymptot. Verhalten



von $\phi(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \, \frac{\rho(\underline{r}')}{|\underline{r}-\underline{r}'|}$ für $r \to \infty$

Methode: Entwickl. des Integranden in Taylorreihe
für $r \gg r'$

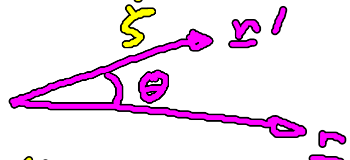
$$G(\underline{r}-\underline{r}') = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \left(\underline{r}'\cdot\underline{\nabla}_r\right)^\ell G(\underline{r})$$

$$\phi(\underline{r}) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \int d^3r' \left(\underline{r}'\cdot\underline{\nabla}_r\right)^\ell G(\underline{r})\,\rho(\underline{r}')$$

explizit mit $G(\underline{r}-\underline{r}') = \frac{1}{4\pi\varepsilon_0|\underline{r}-\underline{r}'|}$:

$$\frac{1}{|\underline{r}-\underline{r}'|} = \left(r^2 - 2\underline{r}\underline{r}' + r'^2\right)^{-1/2} = \frac{1}{r}\Big(1 - 2\frac{r'}{r}\underbrace{\cos\theta}_{\zeta} + \frac{r'^2}{r^2}\Big)^{-1/2}$$

Konvergente Reihe für $r'<r$, $|\zeta|<1$



$$\underbrace{\left(1 - 2\frac{r'}{r}\zeta + \left(\frac{r'}{r}\right)^2\right)^{-1/2}}_{\text{Erzeugende}} = \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell P_\ell(\zeta)$$

$\Rightarrow$ $\underline{\text{Legendre-Polynome}}$ $P_\ell(\zeta)$
(Kugelfunktionen)

Es gilt: $$P_\ell(\zeta) = \frac{1}{\ell!}\left[\frac{\partial^\ell}{\partial t^\ell}\left(1-2t\zeta+t^2\right)^{-1/2}\right]_{t=0}$$

insbesondere
$$P_0(\zeta) = 1$$
$$P_1(\zeta) = \zeta = \cos\theta$$
$$P_2(\zeta) = \frac{1}{2}\left(3\zeta^2 - 1\right) = \frac{1}{4}\left(3\cos 2\theta + 1\right)$$

also

$$\phi(\underline{r}) = \frac{1}{4\pi\epsilon_0}\,\frac{1}{r}\int d^3r'\,\rho(\underline{r}')\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^{l} P_l(\cos\theta)$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty} Q_l\, r^{-l-1}$$

mit $Q_l = \int d^3r'\; r'^{\,l}\rho(\underline{r}')\, P_l(\cos\theta)$ $\quad 2^l$-Pole

Entwicklung nach Potenzen von r !

$\underline{l=0}$: $\phi^{(0)}(\underline{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q_0}{r}$, $Q_0 = \int d^3r'\,\rho(\underline{r}')$ (Gesamtladung)

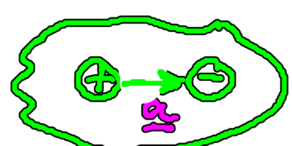
fällt am langsamsten ab! $\quad$ Monopol

$\underline{l=1}$: $\phi^{(1)}(\underline{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{\underline{p}\cdot\underline{r}}{r^3}$

$$Q_1 = \int d^3r'\,\rho(\underline{r}')\underbrace{r'\cos\theta}_{\frac{\underline{r}'\cdot\underline{r}}{r}} = \frac{\underline{p}\cdot\underline{r}}{r}$$

$$\underline{p} := \int d^3r'\,\rho(\underline{r}')\,\underline{r}'$$ Dipolmoment

fällt $\sim \frac{1}{r^2}$



$(Q_0 = 0)$

wichtigster Term für insgesamt neutrale Körper

Beispiel: 2 Punktladung q, $-q$ bei $\underline{r}_1$, $\underline{r}_2$:

$$\rho(\underline{r}') = q\left[\delta(\underline{r}'-\underline{r}_1) - \delta(\underline{r}'-\underline{r}_2)\right]$$

$Q_0 = 0 \,, \quad \underline{p} = q(\underline{r}_1 - \underline{r}_2) = q\underline{a}$

<u>Feld</u> des Dipolpotentials:

$$E_i = -\frac{1}{4\pi\epsilon_0}\,\partial_i \frac{p_k x_k}{r^3} = \frac{1}{4\pi\epsilon_0}\left(\frac{3x_i p_k x_k}{r^5} - \delta_{ik}\frac{p_k}{r^3}\right)$$

(Summ. konv.)

$$\underline{E}(\underline{r}) = \frac{1}{4\pi\epsilon_0}\,\frac{1}{r^5}\left(3(\underline{p}\cdot\underline{r})\underline{r} - r^2\underline{p}\right)$$

$$\sim \frac{1}{r^3} \qquad \text{für } r \to \infty$$

$\underline{\ell = 2}$

$\underline{\ell = 2}$: $\phi^{(2)}(\underline{r}) = \frac{1}{4\pi\epsilon_0}\,\frac{Q_2}{r^3}$

$$Q_2 = \frac{1}{2}\int d^3r'\,\rho(\underline{r}')\,(r')^2\,(3\cos^2\theta - 1)$$

$$= \frac{1}{2}\int d^3r'\,\rho(\underline{r}')\Big(3\,\underbrace{\frac{\underline{r}'\cdot\underline{r}}{r}\,\frac{\underline{r}'\cdot\underline{r}}{r}}_{\frac{x_k' x_k\, x_\ell' x_\ell}{r^2}} - (r')^2\Big)$$

$$= \frac{1}{2r^2}\int d^3r'\,\rho(\underline{r}')\,(\underline{3}\,x_k' x_\ell' - (r')^2\,\delta_{k\ell})\,\overbrace{x_k x_\ell}^{r^2}$$

Q_{kl} Quadrupolmoment

(Spurfrei, symm. Tensor: $\sum_{i=1}^{3} Q_{ii} = \int d^3 r' \rho(\underline{r}')(3r'^2 - 3(r')^2) = 0$)

$\Downarrow$

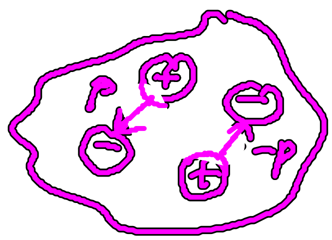
es ex. orthogonale Koord.-trafo auf Diagonalform: $Q_{kl} = 0 \quad (k \neq l)$

$\Rightarrow$ nur 2 unabh. Komp.

$$\phi^{(2)}(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \frac{1}{2r^5} Q_{kl} x_k x_l = \frac{1}{4\pi\varepsilon_0} \frac{\underline{r}\,\underline{\underline{Q}}\,\underline{r}}{2r^5}$$

$$\sim \frac{1}{r^3}$$

Beispiel: 2 entgegengerichtete Dipole:



1.5 Elektrostat. Feldenergie

Kraft $\underline{F}(\underline{r}) = q\underline{E}(\underline{r}) = -q\underline{\nabla}\phi(\underline{r})$

$\Rightarrow$ $V(\underline{r}) = q\phi(\underline{r})$ pot. Energie einer Ladung q im el. Feld $\underline{E}(\underline{r})$

$$W_{ij} = q_i \frac{1}{4\pi\varepsilon_0} \frac{q_j}{|\underline{r}_i - \underline{r}_j|} = W_{ji}$$

Gesamte pot. Energie eines Systems von Ladungen

$$W = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} W_{ij} = \frac{1}{8\pi\varepsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\underline{r}_i - \underline{r}_j|}$$

Kontinuierliche Ladungsverteilung $\rho(\underline{r})$:

$$W = \frac{1}{8\pi\epsilon_0} \int d^3r \int d^3r' \frac{\rho(\underline{r})\,\rho(\underline{r}')}{|\underline{r}-\underline{r}'|}$$

$$\boxed{W = \frac{1}{2} \int d^3r \; \phi(\underline{r})\,\rho(\underline{r})}$$

Mit $\rho = \epsilon_0 \underline{\nabla} \cdot \underline{E}$ folgt:

$$W = \frac{\epsilon_0}{2} \int d^3r \; \phi(\underline{r})\, \underline{\nabla} \cdot \underline{E}$$

$$= \frac{\epsilon_0}{2} \left[\int_{\mathbb{R}^3} d^3r \; \underline{\nabla}(\phi \underline{E}) - \int d^3r \, (\nabla \phi)\, \underline{E}(\underline{r}) \right]$$

$$\overset{\text{gauß}}{=} \frac{\epsilon_0}{2} \left[\underbrace{\int_{S_\infty} d\underline{f} \, (\overset{\sim r^{-1}}{\phi} \; \overset{\sim r^{-2}}{\underline{E}})}_{} + \int d^3r \left(\underline{E}(\underline{r}) \right)^2 \right]$$

$$= \int d^3r \; w(\underline{r})$$

mit $w(\underline{r}) = \frac{\epsilon_0}{2} \left(\underline{E}(\underline{r}) \right)^2$ = Energiedichte des el. Feldes