

Vierstromdichte $j^i := (c\rho, \underline{j}) \quad i=0,1,2,3$

Vierpotenzial $\phi^i := (\phi, c\underline{A})$

Lorentz-Trafo $U^i_k = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \beta = \frac{v}{c}$
 $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

antisymm. Feldtensor $F^{ik} := \partial^i \phi^k - \partial^k \phi^i = -F^{ki}$

$$F^{ik} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{pmatrix}$$

Lorentz-Trafo der Felder

Als Tensor 2. Stufe transformiert sich F^{ik}
wie $\bar{F}^{ik} = U^i_l U^k_m F^{lm}$

Im einzelnen:

$$E^1 = F^{10} = U^1_l U^0_m F^{lm} = -\beta\gamma U^0_m F^{0m}$$

$$+ \gamma U^0_m F^{1m}$$

$$= (\beta\gamma)^2 F^{01} + \gamma^2 F^{10}$$

$$= \underbrace{\gamma^2 (1 - \beta^2)}_1 F^{10} = E^1$$

$$E^2 = F^{12} = U^2_l U^0_m F^{lm} = U^0_m F^{2m} = \gamma F^{20} - \beta\gamma F^{21}$$
$$= \gamma (E^2 - vB^3)$$

$$E^{13} = F^{130} = U^0_{\mu} F^{3\mu} = \gamma F^{30} - \beta \gamma F^{31} \\ = \gamma (E^3 + v B^2)$$

$$B^{11} = \frac{1}{c} F^{132} = \frac{1}{c} U^2_{\nu} U^2_{\mu} F^{3\nu\mu} = \frac{1}{c} F^{32} = B^1$$

$$B^{12} = \frac{1}{c} F^{113} = \frac{1}{c} U^1_{\nu} U^3_{\mu} F^{2\nu\mu} - \frac{1}{c} U^1_{\nu} F^{23} = \gamma (B^2 + \frac{v}{c^2} E^3)$$

$$B^{13} = \gamma (B^3 - \frac{v}{c^2} E^2)$$

Zus. fassung

$E^{11} = E^1$	$B^{11} = B^1$
$E^{12} = \gamma (E^2 - v B^3)$	$B^{12} = \gamma (B^2 + \frac{v}{c^2} E^3)$
$E^{13} = \gamma (E^3 + v B^2)$	$B^{13} = \gamma (B^3 - \frac{v}{c^2} E^2)$

Elektr. \leftrightarrow magn. Felder

Umrechnung

$$\tilde{\phi}^i = \phi^i + \partial^i \varphi \quad \text{Eich fkt. } \varphi \text{ (Lorentz-Skala)}$$

$$\begin{aligned} \rightarrow \tilde{F}^{ik} &= \partial^i \tilde{\phi}^k - \partial^k \tilde{\phi}^i = \partial^i (\phi^k + \partial^k \varphi) - \partial^k (\phi^i + \partial^i \varphi) \\ &= \underbrace{\partial^i \phi^k - \partial^k \phi^i}_{= F^{ik}} + \underbrace{\partial^i \partial^k \varphi - \partial^k \partial^i \varphi}_{= 0} = F^{ik} \end{aligned}$$

F^{ik} ist invariant!

Homogene Maxwell-Gleichungen

$$(1) \nabla \cdot \underline{B} = \partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3 = 0$$

$$\Rightarrow \partial_1 F^{32} + \partial_2 F^{13} + \partial_3 F^{21} = 0$$

$$\text{mit } \partial_1 = -\partial^1, \dots, F^{32} = -F^{23} \text{ folgt}$$

$\partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12} = 0$	zykl (123)
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$$(2) \quad \nabla_x \underline{E} + \frac{\partial}{\partial t} \underline{B} = 0$$

$$1. \text{ Komp. : } \partial_2 E^3 - \partial_3 E^2 + \frac{\partial}{\partial t} B^1 = 0$$

$$\rightarrow \partial_2 F^{30} - \partial_3 F^{20} + \partial_0 F^{32} = 0$$

$$\text{mit } \partial_0 = \partial^0, \partial_2 = -\partial^2, \partial_3 = -\partial^3, F^{32} = -F^{23}, F^{20} = -F^{02}$$

$$\partial^0 F^{23} + \partial^2 F^{30} + \partial^3 F^{02} = 0$$

zykl. (023)

zykl. Permutation $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ mit $F^{ik} = -F^{ki}$

$$\partial^0 F^{13} + \partial^3 F^{01} + \partial^1 F^{30} = 0$$

$$\partial^0 F^{12} + \partial^1 F^{20} + \partial^2 F^{01} = 0$$

zykl. (013)

zykl. (012)

Zus. fassung der homog. Maxwellgl.:

$$\epsilon_{iklm} \partial^k F^{lm} = 0$$

oder

$$\epsilon^{iklm} \partial_k F_{lm} = 0 \quad \text{4-Rotat.}^2$$

mit $\epsilon_{iklm} = \begin{cases} 1 & \text{wenn } (iklm) = \text{gerade Permutation von } (0123) \\ -1 & \text{wenn } (iklm) = \text{ungerade Perm.} \\ 0 & \text{sonst} \end{cases}$

4-dim. Levi-Civita-Tensor

(Rot. eines 3-Vektor $(\nabla \times \underline{a})_i = \epsilon^{ikl} \partial_k a_l$)
Pseudovektor = axialer Vektor
Bem.: (i) ϵ^{iklm} ist total antisymmetrisch.

(ii) ϵ^{iklm} transformiert sich unter Lorentz-Transf. wie ein Pseudotensor

$$\epsilon'^{iklm} = (\det U) U^i_n U^k_p U^l_q U^m_r \epsilon^{npqr}$$

damit $\epsilon'^{iklm} = \pm 1 \epsilon^{iklm} !$

Inhomog. Maxwell-Gln. (im Vakuum)

$$(3) \quad \epsilon_0 \nabla \cdot \underline{E} = \rho$$

$$\partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3 = \frac{1}{\epsilon_0} \rho$$

$$\partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \frac{1}{\epsilon_0} j^0$$

$$\partial_i F^{i0} = \frac{1}{\epsilon_0} j^0$$

$$(4) \quad \nabla \times \underline{E} = \mu_0 \underline{j} + \epsilon_0 \mu_0 \frac{\partial \underline{E}}{\partial t}$$

$$1. \text{ Komp. : } \partial_2 E^3 - \partial_3 E^2 = \mu_0 j^1 + \epsilon_0 \mu_0 \frac{\partial E^1}{\partial t} \quad | \cdot c$$

$$\mu_0 c = \frac{1}{\epsilon_0 c} : \partial_2 F^{21} - \partial_3 F^{31} = \frac{1}{\epsilon_0 c} j^1 + \partial_0 F^{10}$$

$$\partial_1 F^{10} + \partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} = \frac{1}{\epsilon_0 c} j^1$$

$$\partial_i F^{i1} = \frac{1}{\epsilon_0 c} j^1$$

analog für 2. u. 3. Komp. !

Zus.fassung der inhomog. Maxwellgln. :

$$\partial_i F^{ik} = \frac{1}{\epsilon_0 c} j^k$$

"4-Divergenz"

Bem. : (i) Die homog. Maxwell-Gln. sind durch den Potenzialansatz

$$F_{lm} = \partial_l \phi_m - \partial_m \phi_l$$

automatisch erfüllt :

$$\epsilon^{iklm} \partial_k F_{lm} = \underbrace{\epsilon^{iklm} \partial_k \partial_l \phi_m} - \underbrace{\epsilon^{iklm} \partial_k \partial_m \phi_l} = 0$$

$$\begin{aligned}
 & \epsilon^{iklm} \partial_k \partial_l \phi_m = -\epsilon^{ilk m} \partial_l \partial_k \phi_m \\
 & \dots \dots \dots \\
 & \text{wegen Antisym. von } \epsilon^{iklm} = -\epsilon^{ilk m} \partial_l \partial_k \phi_m \\
 & \dots \dots \dots
 \end{aligned}$$

(ii) Aus den inhomog. Maxwell-Gln.

$$\partial_i F^{ik} = \partial_i \partial^i \phi^k - \partial_i \partial^k \phi^i = \frac{1}{\epsilon_0 c} j^k$$

folgt mit der Lorenz-Eichung $\partial_i \phi^i = 0$:

$$\partial_i \partial^k \phi^i = \partial^k \underbrace{\partial_i \phi^i}_0 = 0$$

also

$$\partial_i \partial^i \phi^k = \frac{1}{\epsilon_0 c} j^k$$

inhomog. Wellengl.

(iii) Die Maxwell-Gln.

$$\begin{aligned}
 \epsilon^{iklm} \partial_k F_{lm} &= 0 \\
 \partial_i F^{ik} &= \frac{1}{\epsilon_0 c} j^k
 \end{aligned}$$

sind Lorentz-kovariant, weil sie durch 4-Vektoren ausgedrückt sind
 (pseudo-4-Vektor stört nicht, da rechte Seite = 0)