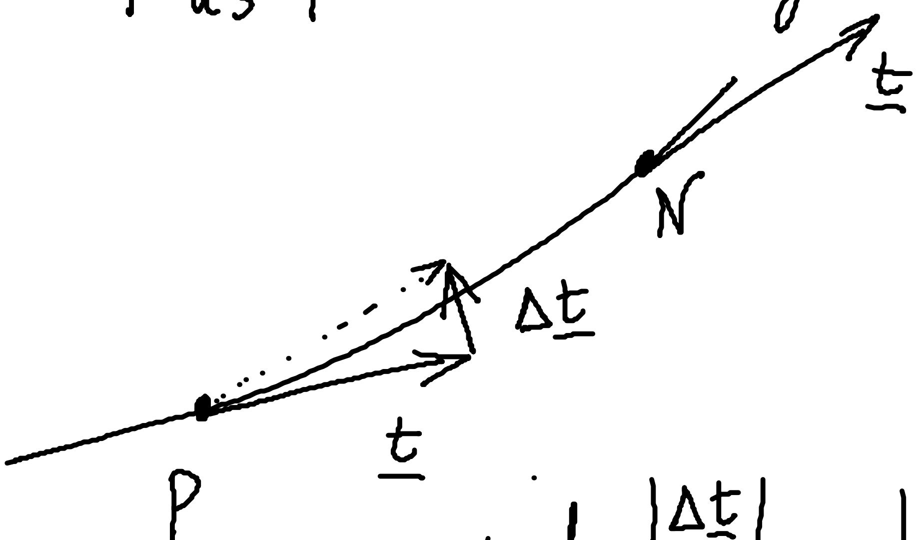
12.6.08

$$\underline{t}(s) = \frac{d}{ds}\,\underset{\sim}{\underline{r}}(s) \qquad ds = |v(t)|\,dt$$

Tangentialvektor $\qquad$ s Bogenlänge

$$\underline{n}(s) = \frac{\frac{d\underline{t}}{ds}}{\left|\frac{d\underline{t}}{ds}\right|},$$

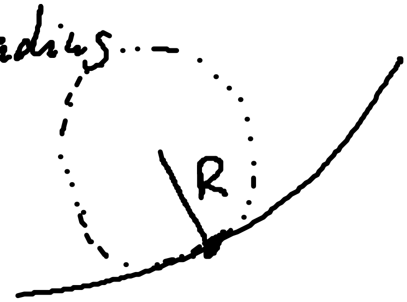
$$K \equiv \left|\frac{d\underline{t}}{ds}\right| \qquad \text{Krümmung}$$



$$K = \lim_{\overset{\frown}{PN}\to 0} \frac{|\Delta \underline{t}|}{|\overset{\frown}{PN}|} = \left|\frac{dt(s)}{ds}\right|$$

$$= \left|\frac{d^2 r(s)}{ds^2}\right| \qquad \text{Krümmung}$$

$$R \equiv \frac{1}{K} \qquad \text{Krümmungsradius}$$



Geschwindigkeit: $$\underline{v}(t) = \frac{d\underline{r}(t)}{dt} = \underbrace{\frac{d\underline{\tilde{r}}(s)}{ds}}_{\underline{t}(s)} \frac{ds}{dt}$$

Merke $ds = |v(t)|\,dt$, damit

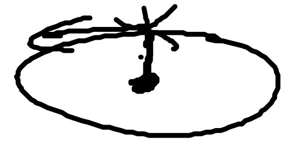
$$\underline{v}(t) = |v|\,\underline{t}\,, \quad \underline{t} \text{ Tangentialvektor}$$

$$\underline{a}(t) = \frac{d}{dt}\underline{v}(t) =$$

$$= \frac{d}{dt}\Big[\underbrace{v(t)}_{=|\underline{v}(t)|}\,\underline{t}(s(t))\Big] = \dot{v}(t)\,\underline{t}(s(t)) + v(t)\frac{d}{dt}\underline{t}(s(t))$$

$$= \dot{v}(t)\,\underline{t}(s(t)) + v(t)\underbrace{\frac{d\underline{t}}{ds}}_{\underline{n}\cdot\underbrace{\left|\frac{d\underline{t}}{ds}\right|}_{K = 1/R}}\underbrace{\frac{ds}{dt}}_{v(t)}$$

(Definition von $\underline{n}$)



$$= \underbrace{\dot{v}(t)\,\underline{t}(s(t))}_{\text{Tangentialbeschl.}} + \underbrace{\frac{v^2(t)}{R}\,\underline{n}(s(t))}_{\text{Zentripetalbeschl.}}$$

$$\underline{n} = \frac{\dot{t}}{|t|}$$

$\underline{t}$ Tangentialvektor

$\underline{n}$ Normalenvektor

$\underline{b} = \underline{t} \times \underline{n}$ Binormalenvektor

$$\frac{d\underline{b}}{ds} = \underbrace{\cancel{\dot{t}(s)}}_{\propto\, \underline{n}} \times n + \underline{t} \times \dot{n}(s)$$

proportional zu

Also $\frac{db}{ds} \perp \underline{t}$, außerdem

$$1 = \underline{b} \cdot \underline{b} \Rightarrow d/ds\, 1 = 0 = 2\underline{\dot{b}}\underline{b}$$

Also $\frac{d\underline{b}}{ds} \perp \underline{t}$ und $\underline{b}$

$$\boxed{\frac{d\underline{b}}{ds} = -\tau(s)\,\overset{\underline{n}}{\cancel{\underline{b}}}(s), \quad \tau(s) \text{ Torsion}}$$

$$\Longrightarrow \quad \boxed{\begin{aligned} \frac{d}{ds}\underline{b} &= -\tau\,\cancel{\underline{b}}\;\underline{n} && \text{(Definition)} \\ \frac{d}{ds}\underline{t} &= K\,\underline{n} && \text{(Definition)} \\ \frac{d}{ds}\underline{n} &= -K\underline{t} + \tau\underline{b} && \text{(AUFGABE)} \end{aligned}}$$

$\left.\begin{matrix} \tau(s) \\ K(s) \end{matrix}\right\}$ bestimmen die Form der Kurve

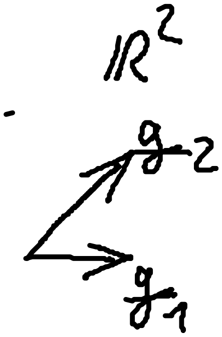
Frenetschen Gleichungen

Krummlinige Koordinaten

Kovariante und kontravariante Basis

Lit.: E. Klingbeil: Tensorrechnung für Ingenieure.

Sei $\underline{g}_i$ eine Basis des $\mathbb{R}^n$ z.B.



Wir konstruieren Basis

$\underline{g}^i$ (Index oben) mit

$$\boxed{\underline{g}_i \cdot \underline{g}^j = \delta_i^{\ j}}$$ Kronecker-δ

$\underline{g}_i$ kovariante Basis

$\underline{g}^j$ kontravariante Basis

$g_{ij} \equiv \underline{g}_i \cdot \underline{g}_j$ kovariante Metrikkoeff.

$g^{ij} \equiv \underline{g}^i \cdot \underline{g}^j$ kontravariante "

A^{ij} gesucht: $\underline{g}^i = \sum A^{ij} \underline{g}_j = A^{ij} \underline{g}_j \;\Big/ \cdot \underline{g}^k$

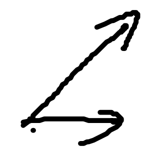
(Transformations-Matrix) (Einsteinsche Summationskonvention)

$$\Rightarrow \quad \underbrace{\underline{g}^i \underline{g}^k}_{g^{ik}} = A^{ij} \underbrace{\underline{g}_j \underline{g}^k}_{\delta_j^{\ k}}$$

$$\Rightarrow \quad \underline{g^{ik}} = A^{ij} \delta_j^{\ k} = \underline{A^{ik}}$$

Entsprechend $g^i = g^{ij} g_j$

$g_k = g_{kj} g^j$

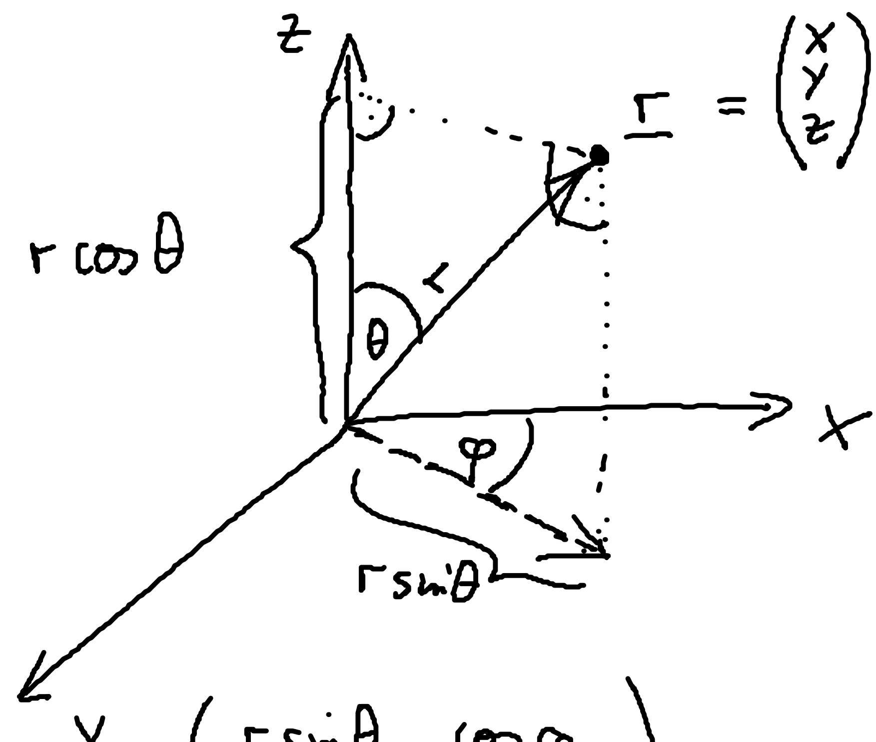


$$\underbrace{g^l g_k}_{\delta^l_k} = g_{kj} \underbrace{g^j g^l}_{g^{jl}} \Rightarrow \delta^l_k = g_{kj} g^{jl}$$

$$\underline{\mathbb{1}} = G_{-}\, G^{-}$$

g_{ij}, g^{ij} sind Matrizen, die invers zueinander.

Polarkoordinaten in d=3 (sphärische PKO)



$$\Big(r \sin\theta \cos\varphi \Big.$$

$$\underline{r} = \begin{pmatrix} \\ r \sin\theta \;\; \sin\varphi \\ r\cos\theta \end{pmatrix}$$

günstig für radialsymmetrische Probleme, z.B. Coulombpotential

$$\phi(x,y,z) = \frac{e}{4\pi\varepsilon_0} \frac{1}{\sqrt{x^2+y^2+z^2}}$$

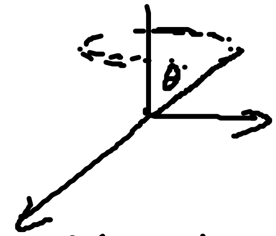
$$= \frac{e}{4\pi\varepsilon_0} \frac{1}{r}$$

$\theta = \text{const}$
$\varphi = \text{const}$ in Radialrichtung

$$\theta, \varphi = \text{const}: \quad \frac{d}{dt} \begin{pmatrix} r(t) \cos\varphi \sin\theta \\ r(t) \sin\varphi \sin\theta \\ r(t) \cos\theta \end{pmatrix} =$$

$$= \dot r(t) \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix} = \text{Geschw. in Radialrichtung}$$

$$\equiv \underline{e}_r$$ Einheitsvektor



$\theta, r = \text{const}$:

$$\frac{d}{dt}\begin{pmatrix} r\cos\varphi(t)\,\sin\theta \\ r\sin\varphi(t)\,\sin\theta \\ r\cos\theta \end{pmatrix} = r\sin\theta\cdot\dot{\varphi}\cdot\underbrace{\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}}_{\underline{e}_\varphi}$$

$$\underline{e}_r\cdot\underline{e}_\varphi = 0$$

$r, \varphi = \text{const}$:

$$\frac{d}{dt}\begin{pmatrix} r\cos\varphi\,\sin\theta(t) \\ r\sin\varphi\,\sin\theta(t) \\ r\cos\theta(t) \end{pmatrix} = r\dot{\theta}(t)\underbrace{\begin{pmatrix} \cos\varphi\,\cos\theta \\ \sin\varphi\,\cos\theta \\ -\sin\theta \end{pmatrix}}_{\underline{e}_\theta} ; \quad \begin{array}{l} \underline{e}_\theta\cdot\underline{e}_\varphi = 0 \\ \underline{e}_\theta\cdot\underline{e}_r = 0 \end{array}$$

Wir erhalten 3 lokale, orthogonale Basisvektoren

$$\underline{e}_r,\ \underline{e}_\varphi,\ \underline{e}_\theta$$

Damit haben wir die Geschwindigkeit in PKO

$$\underline{V} = \underline{V}(t) = \frac{d}{dt}\underline{r}(t) =$$

$$= \dot{r}\,\underline{e}_r + r\dot{\varphi}\sin\theta\,\underline{e}_\varphi + r\dot{\theta}\,\underline{e}_\theta$$

$L = T - V = T =$ kinetische Energie

$\Updownarrow$ Lagrange-Fkt

$$= \frac{1}{2} m \dot{\underline{r}}^2 = \frac{m}{2} \underbrace{\left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\varphi}^2 \right)}$$

Entsprechend die Bogenlänge

$$ds = |\underline{v}(t)| \, dt = \underbrace{\left(\quad \right)}^{1/2} dt$$

Notation: $ds^2 = \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\varphi}^2 \right) dt^2$; $\quad \dot{r}^2 = \left(\frac{dr}{dt} \right)^2$

$$= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

Allgemein: $$ds^2 = \sum_{\alpha,\beta=1}^{3} g_{\alpha\beta}(x^\nu) \, dx^\alpha dx^\beta$$

metrische Fundamentalform.

Beliebige Basis, Metrik-Tensor

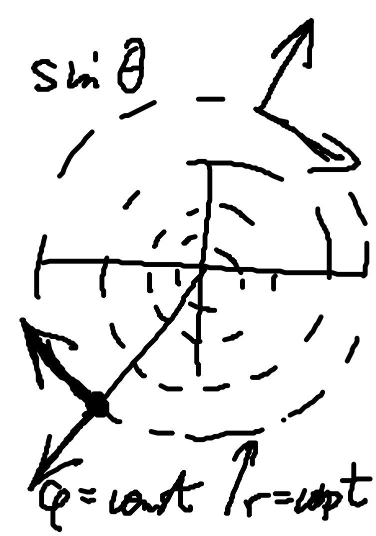
$$\underline{r} = \sum_k x^k(u^1, u^2, \ldots u^d) \, \underline{e}_k$$

z.B. $\quad x = r \cos\varphi \sin\theta$

$r = u^1$

$\varphi = u^2$

$\theta = u^3$



$$\underset{\sim}{g_i} \equiv \frac{\partial \underline{r}}{\partial u^i} =$$

kat. Basis

$$= \frac{\partial}{\partial u^i} \sum_k x^k \underline{e}_k = \sum_k \frac{\partial x^k}{\partial u^i} \underline{e}_k$$

$\{\underline{g}_i\}$ ist Basis falls $\det(g_1, \dots, g_d) \neq 0$

Geschwindigkeit $\underline{v} = \dot{\underline{r}} = \sum_{k,i} \frac{\partial x^k}{\partial u^i} \dot{u}^i \underline{e}_k = \underbrace{\frac{\partial x^k}{\partial u^i}}\dot{u}^i \underbrace{\underline{e}_k}$

$= \dot{u}^i \underline{g}_i$

Bogenlänge

$$ds = |\underline{v}|\,dt = \left(\sum_i \dot{u}^i \underline{g}_i \sum_j \dot{u}^j \underline{g}_j\right)^{1/2} dt$$

$$= \Big(\sum_{i,j} \dot{u}^i \dot{u}^j \underbrace{\underline{g}_i \underline{g}_j}_{g_{ij}}\Big)^{1/2} dt$$

g_{ij} metrischer Tensor

$$ds^2 = g_{ij}\, du^i\, du^j$$

$$g_{ij} = \underline{g}_i \cdot \underline{g}_j = \frac{\partial \underline{r}}{\partial u^i} \cdot \frac{\partial \underline{r}}{\partial u^j}$$