

12.6.08

$$\underline{t}(s) = \frac{d}{ds} \underline{r}(s)$$

Tangentenvektor

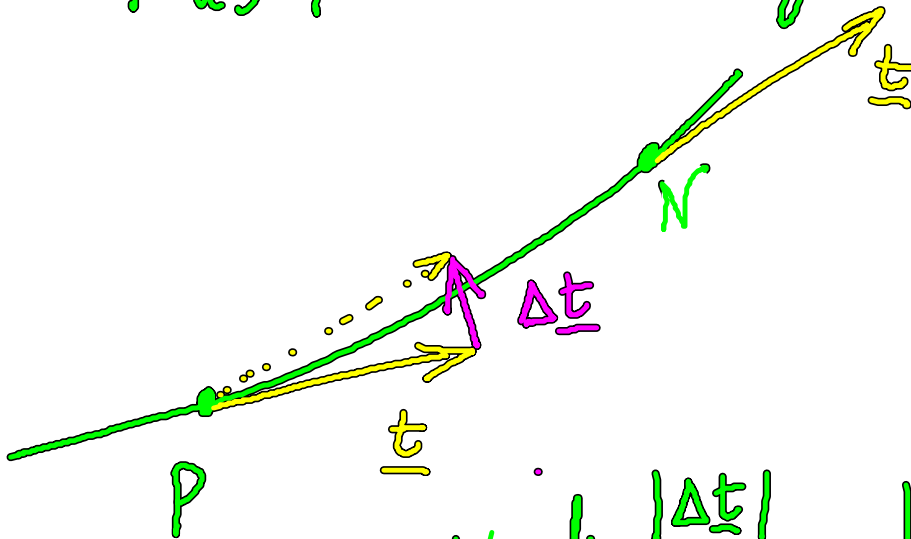
$$ds = |v(t)| dt$$

s Bogenlänge

$$\underline{n}(s) = \frac{\frac{d\underline{t}}{ds}}{\left| \frac{d\underline{t}}{ds} \right|},$$

$$K \equiv \left| \frac{d\underline{t}}{ds} \right|$$

Krümmung

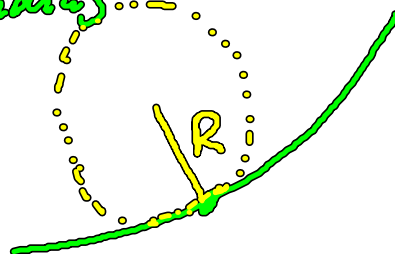


$$K = \lim_{PN \rightarrow 0} \frac{|\Delta \underline{t}|}{|\widehat{PN}|} = \left| \frac{d\underline{t}(s)}{ds} \right|$$

$$= \left| \frac{d^2 \underline{r}(s)}{ds^2} \right| \quad \text{Krümmung}$$

$$R \equiv \frac{1}{K}$$

Krümmungsradius



geschwindigkeit: $\underline{v}(t) = \frac{d\underline{r}(t)}{dt} = \frac{d\underline{r}(s)}{ds} \frac{ds}{dt}$

Merke $ds = |v(t)| dt$, damit $\underline{t}(s)$

$\underline{v}(t) = |v| \underline{t}$, \underline{t} Tangentialvektor



$\underline{a}(t) = \frac{d}{dt} \underline{v}(t) =$

$= \frac{d}{dt} [\underbrace{|v(t)|}_{\text{speed}} \underline{t}(s(t))] = \dot{v}(t) \underline{t}(s(t)) + v(t) \frac{d}{dt} \underline{t}(s(t))$

$= \dot{v}(t) \underline{t}(s(t)) + v(t) \frac{d\underline{t}}{ds} \frac{ds}{dt}$

(Definition von \underline{n})

$\underline{n} \cdot \left| \frac{d\underline{t}}{ds} \right|$
 $\kappa = 1/R$



$= \underbrace{\dot{v}(t) \underline{t}(s(t))}_{\text{Tangentialbesch.}} + \underbrace{\frac{v^2(t)}{R} \underline{n}(s(t))}_{\text{Zentripetalbesch.}}$

Tangentialbesch.

Zentripetalbesch.

$$\underline{n} = \frac{\dot{\underline{t}}}{|\dot{\underline{t}}|}$$

\underline{t} Tangentialvektor

\underline{n} Normalenvektor

$\underline{b} = \underline{t} \times \underline{n}$ Binormalenvektor

$$\frac{d\underline{b}}{ds} = \dot{\underline{t}}(s) \times \underline{n} + \underline{t} \times \dot{\underline{n}}(s)$$

$\dot{\underline{t}} \propto \underline{n}$
 \uparrow proportional zu



Also $\frac{d\underline{b}}{ds} \perp \underline{t}$, außerdem

$$1 = \underline{b} \cdot \underline{b} \Rightarrow \frac{d}{ds} 1 = 0 = 2 \underline{\dot{b}} \cdot \underline{b}$$

Mit $\frac{db}{ds} \perp \underline{t}$ und \underline{b}

$$\frac{db}{ds} = -\tau(s) \underline{A}(s), \quad \tau(s) \text{ Torsion}$$

$$\Rightarrow \left. \begin{aligned} \frac{d}{ds} \underline{b} &= -\tau \underline{A} \\ \frac{d}{ds} \underline{t} &= K \underline{n} \\ \frac{d}{ds} \underline{n} &= -K \underline{t} + \tau \underline{b} \end{aligned} \right\} \begin{array}{l} \text{(Definition)} \\ \text{(Definition)} \\ \text{(AUFGABE)} \end{array}$$

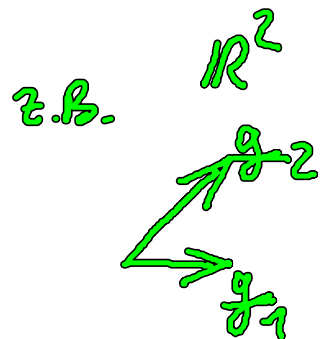
$\tau(s)$
 $K(s)$ } bestimmen die Form
der Kurve
Frenetschen Gleichungen

Krummlinige Koordinaten

Kovariante und kontravariante Basis

Lit. E. Klingbeil : Tensorrechnung für
Ingenieure.

Sei g_i eine Basis des \mathbb{R}^n
Wir konstruieren Basis



$$g^i \text{ (Index oben) mit } \boxed{g_i \cdot g^j = \delta_i^j} \leftarrow \text{Kronecker-}\delta$$

g_i kovariante Basis

g^j kontravariante Basis

$g_{ij} \equiv g_i \cdot g_j$ kovariante Metrikkoef.

$g^{ij} \equiv g^i \cdot g^j$ kontravariante "

A^{ij} gesucht: $g^i = \sum A^{ij} g_j = A^{ij} g_j / g^k$

(Transformationsmatrix) (Einsteinische Summenkonvention)

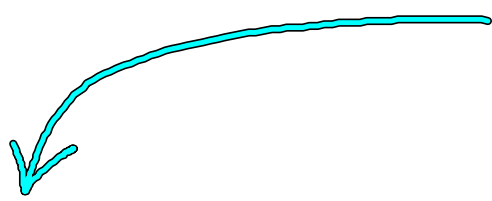
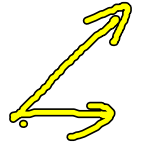
$$\Rightarrow \underbrace{g^i g^k}_{g^{ik}} = A^{ij} \underbrace{g_j g^k}_{\delta_j^k}$$

$$\Rightarrow \underbrace{g^{ik}} = A^{ij} \delta_j^k = \underbrace{A^{ik}}$$

Entsprechend

$$g^i = g^{ij} g_j$$

$$g_k = g_{kj} g^j$$

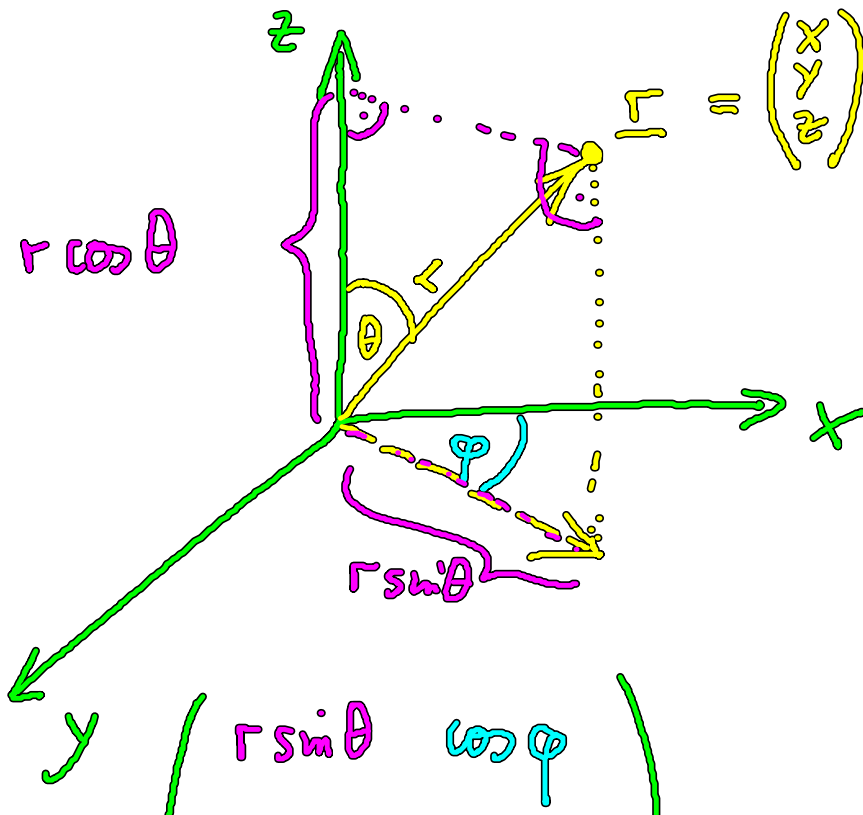


$$\underbrace{g^l g_k}_\delta^l_k = g_{kj} \underbrace{g^j g^l}_{g^{jl}} \Rightarrow \delta^l_k = g_{kj} g^{jl}$$

$\mathbb{1} = G G^{-1}$

g_{ij} , g^{ij} sind Matrizen, die
invers zueinander.

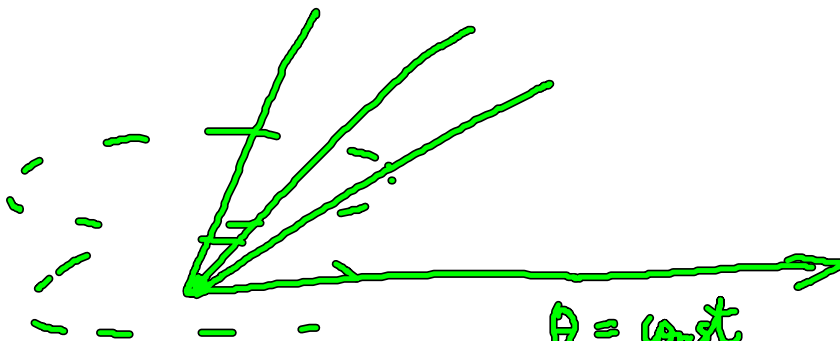
Polarkoordinaten in $d=3$ (sphärische PKO)



$$\underline{r} = \begin{pmatrix} r \sin \theta \sin \varphi \\ r \cos \theta \end{pmatrix}$$

genügend für radialsymmetrische Probleme,
 z.B. Coulombpotential

$$\begin{aligned} \phi(x, y, z) &= \frac{e}{4\pi \epsilon_0} \frac{1}{\sqrt{x^2 + y^2 + z^2}} \\ &= \frac{e}{4\pi \epsilon_0} \frac{1}{r} \end{aligned}$$



$\theta = \text{const}$
 $\varphi = \text{const}$ in Radialrichtung

$$\theta, \varphi = \text{const} : \quad \frac{d}{dt} \begin{pmatrix} r(t) \cos \varphi \sin \theta \\ r(t) \sin \varphi \sin \theta \\ r(t) \cos \theta \end{pmatrix} =$$

$$= r(t) \begin{pmatrix} \cos \varphi \sin \theta \\ \sin \varphi \sin \theta \\ \cos \theta \end{pmatrix} = \text{fester in Radialrichtung}$$

$$= \underline{e}_r$$

Einheitsvektor



$\theta, r = \text{const}$:

$$\frac{d}{dt} \begin{pmatrix} r \cos \varphi(t) \sin \theta \\ r \sin \varphi(t) \sin \theta \\ r \cos \theta \end{pmatrix} = r \sin \theta \cdot \dot{\varphi} \cdot \begin{pmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{pmatrix}$$

$$\underline{e}_r \cdot \underline{e}_\varphi = 0$$

$$r, \varphi = \text{const} : \frac{d}{dt} \begin{pmatrix} r \cos \varphi \sin \theta(t) \\ r \sin \varphi \sin \theta(t) \\ r \cos \theta(t) \end{pmatrix}$$

$$= r \dot{\theta}(t) \begin{pmatrix} \cos \varphi \cos \theta \\ \sin \varphi \cos \theta \\ -\sin \theta \end{pmatrix}$$

$$\begin{aligned} \underline{e}_\theta \cdot \underline{e}_\varphi &= 0 \\ \underline{e}_\theta \cdot \underline{e}_r &= 0 \end{aligned}$$

\underline{e}_θ

Wir erhalten 3 lokale, orthogonale Basisvektoren

$$\underline{e}_r, \underline{e}_\varphi, \underline{e}_\theta$$

Damit haben wir die Geschwindigkeit in PKD

$$\underline{v} = \underline{v}(t) = \frac{d}{dt} \underline{r}(t) =$$

$$= \dot{r} \underline{e}_r + r \dot{\varphi} \sin \theta \underline{e}_\varphi + r \dot{\theta} \underline{e}_\theta$$

$$L = T - V = T = \text{kinetische Energie}$$

↑ Lagrange-Fkt

$$= \frac{1}{2} m \underline{v}^2 = \frac{m}{2} \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\varphi}^2 \right)$$

Entsprechend die Bogenlänge

$$ds = |\underline{v}(t)| dt = \left(\quad \right)^{1/2} dt$$

$$\text{Notation: } ds^2 = \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\varphi}^2 \right) dt^2;$$

$$= dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$$

$$\dot{r}^2 = \left(\frac{dr}{dt} \right)^2$$

$$\text{Allgemein: } ds^2 = \sum_{\alpha, \beta=1}^3 g_{\alpha\beta}(x^\nu) dx^\alpha dx^\beta$$

metrische Fundamentalfonn.

Beliebige Basis, Metrik-Tensor

$$\underline{r} = \sum_k x^k \binom{1 \ 2 \ d}{\mu_1 \ \mu_2 \ \dots \ \mu_k} \underline{e}_k$$

z.B.

$$x = r \cos \varphi \sin \theta$$

$$r = \mu^1$$

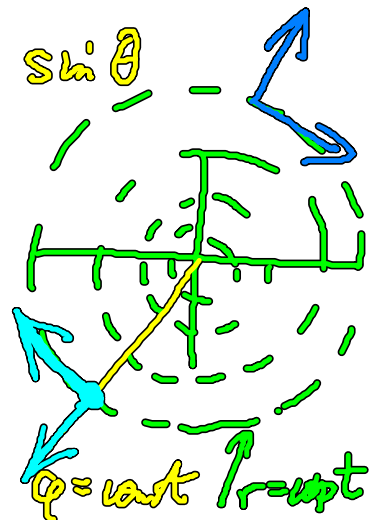
$$\varphi = \mu^2$$

$$\theta = \mu^3$$

kart. Basis

$$\underline{g}_i = \frac{\partial \underline{r}}{\partial \mu^i} =$$

$$= \frac{\partial}{\partial \mu^i} \sum_k x^k \underline{e}_k = \sum_k \frac{\partial x^k}{\partial \mu^i} \underline{e}_k$$



$\{g_i\}$ ist Basis falls $\det(g_1, \dots, g_d) \neq 0$

Geschwindigkeit $\underline{v} = \underline{\dot{\Gamma}} = \sum_{k,i} \frac{\partial x^k}{\partial u^i} \dot{u}^i \underline{e}_k = \underbrace{\frac{\partial x^k}{\partial u^i} \dot{u}^i}_{\dot{u}^i g_i} \underline{e}_k$

Bogenlänge
 $ds = |\underline{v}| dt = \left(\sum_i \dot{u}^i g_i \cdot \sum_j \dot{u}^j g_j \right)^{1/2} dt = \left(\sum_{ij} \dot{u}^i \dot{u}^j \underbrace{g_i g_j}_{g_{ij}} \right)^{1/2} dt$

$ds^2 = g_{ij} du^i du^j$ g_{ij} metrischer Tensor

$$g_{ij} = g_i \cdot g_j = \frac{\partial \underline{r}}{\partial u^i} \cdot \frac{\partial \underline{r}}{\partial u^j}$$