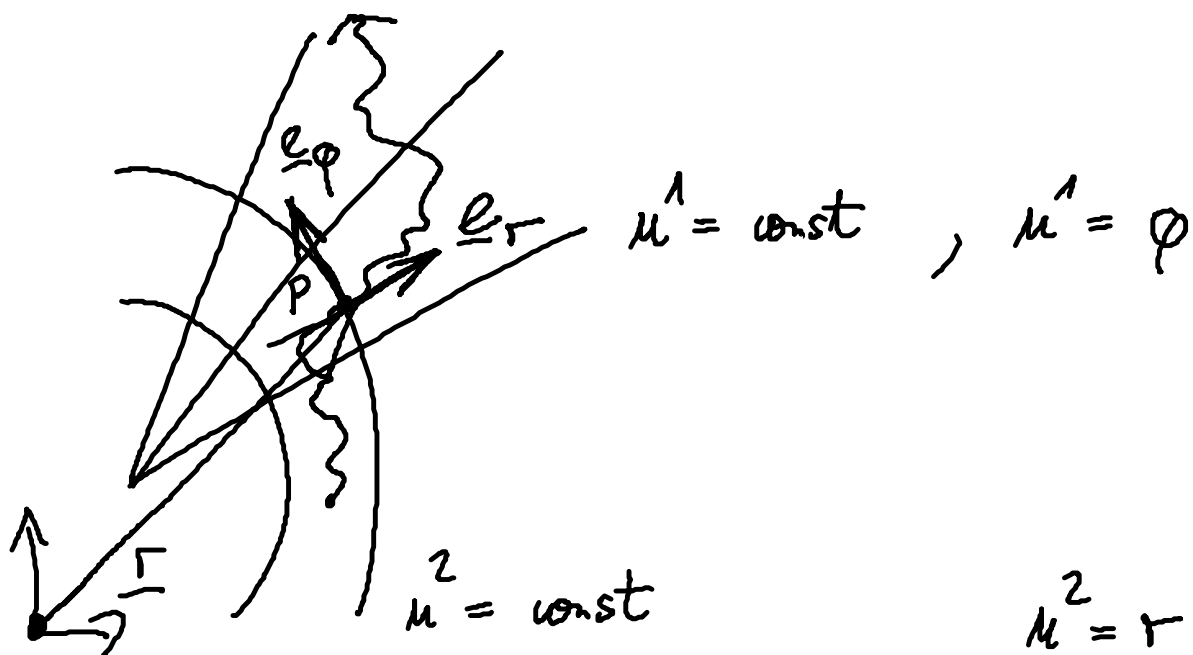


19.6.

19.6.



$$g_i \equiv \frac{\partial \underline{r}}{\partial u^i} = \sum_{k=1}^d \frac{\partial x^k}{\partial u^i} \underline{e}_k \equiv \frac{\partial x^k}{\partial u^i} \underline{e}_k$$

$$\underline{e}_k = \sum_j \frac{\partial u^j}{\partial x^k} g_j \equiv \frac{\partial u^j}{\partial x^k} g_j$$

feschwindigkeit $\underline{v} = \dot{\underline{r}} = \frac{\partial x^k}{\partial u^i} \dot{u}^i \underline{e}_k$
 $= \dot{u}^i g_i$

Beschleunigung

$$\underline{a} = \dot{\underline{v}} = \frac{d}{dt} \dot{u}^i g_i = \ddot{u}^i g_i + \dot{u}^i \dot{g}_i$$

$$\begin{aligned} \dot{g}_i &= \frac{d}{dt} g_i(u^1, \dots, u^d) = \frac{\partial}{\partial u^j} g_i \frac{d}{dt} u^j \\ &= \underbrace{\frac{\partial}{\partial u^j} g_i}_{\Gamma_{ij}^l} \dot{u}^j = \sum_{l,j} \Gamma_{ij}^l g_l \dot{u}^j \end{aligned}$$

Γ_{ij}^l heißen Christoffel-Symbole

Aus der Definition

$$\frac{\partial}{\partial u^j} g_i = \frac{\partial}{\partial u^j} \underbrace{\frac{\partial x^k}{\partial u^i}}_{g_i} e_k = \frac{\partial^2 x^k}{\partial u^j \partial u^i} e_k$$

$$= \underbrace{\frac{\partial^2 x^k}{\partial u^j \partial u^i} \frac{\partial u^l}{\partial x^k}}_{\Gamma_{ij}^l} g^l$$

Also

$$\boxed{\Gamma_{ij}^l = \sum_{k=1}^d \frac{\partial^2 x^k}{\partial u^j \partial u^i} \frac{\partial u^l}{\partial x^k}}$$

Beschleunigung

$$\begin{aligned} \underline{\ddot{\Gamma}} &= \ddot{u}^i g_i + \dot{u}^i \dot{g}_i \\ &= \sum_l \ddot{u}^l g_l + \dot{u}^i \Gamma_{ij}^l g_l \dot{u}^j \\ &= \sum_l \left\{ \ddot{u}^l + \dot{u}^i \dot{u}^j \Gamma_{ij}^l \right\} g_l \end{aligned}$$

z.B. Newton $\underline{F} = m \underline{a}$ mit $\underline{F} = \sum_l f^l \underline{g}_l$

$$\Rightarrow f^l = m \left(\ddot{u}^l + \dot{u}^i \dot{u}^j \Gamma_{ij}^l \right)$$

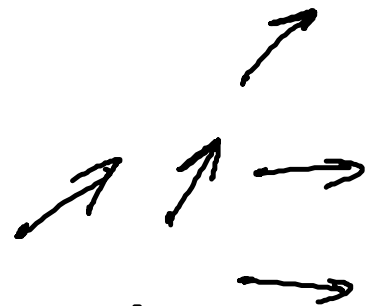
S. PKO $u^1 = r$ $l = 1, 2, 3$
 $u^2 = \varphi$
 $u^3 = \theta$

Literatur: Klingbeil: „Tensorenrechnung für Ingenieure“
Stephani: „Allgemeine Relativitätstheorie“

V Vektoranalysis

Def:

Abbildung $\mathbb{R}^n \rightarrow \mathbb{R}^m$, $n, m \in \mathbb{N}$

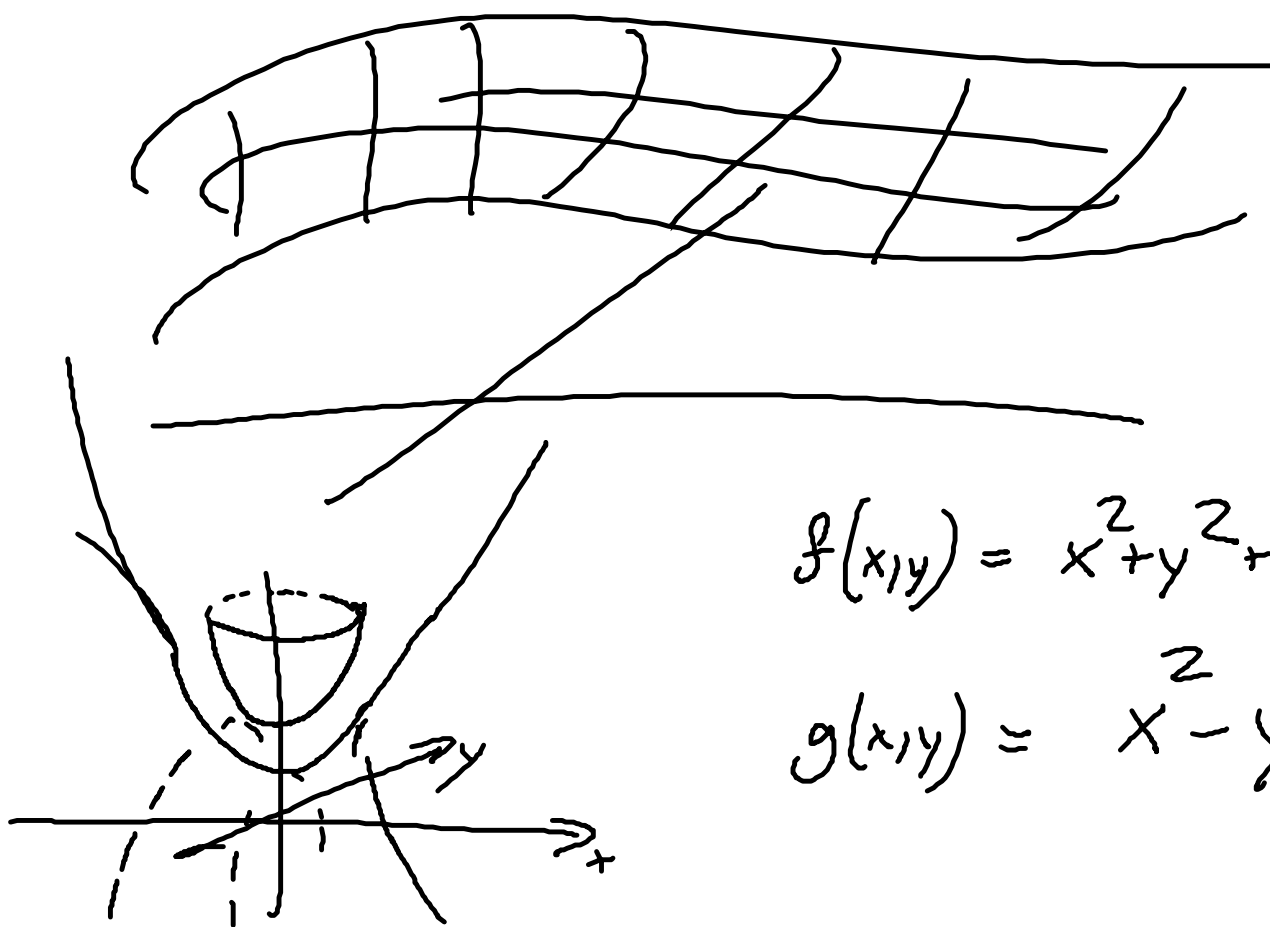


$\underline{x} = (x_1, \dots, x_n)^T \mapsto \underline{f}(\underline{x}) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$
heißt Vektorfeld.

z.B. Elektrisches Feld $\underline{E}(\underline{x}), m=3$ $\underline{E} = \vec{E} = \mathbf{E}$
Temperatur $T(\underline{x}), m=1$ (fett gedruckt).

Bei $m=1$ spricht man von einem
Skalarfeld.

Skizze: Skalarfeld $f(x_1, \dots, x_n)$:



$$f(x,y) = x^2 + y^2 + 1$$

$$g(x,y) = x^2 - y^2$$

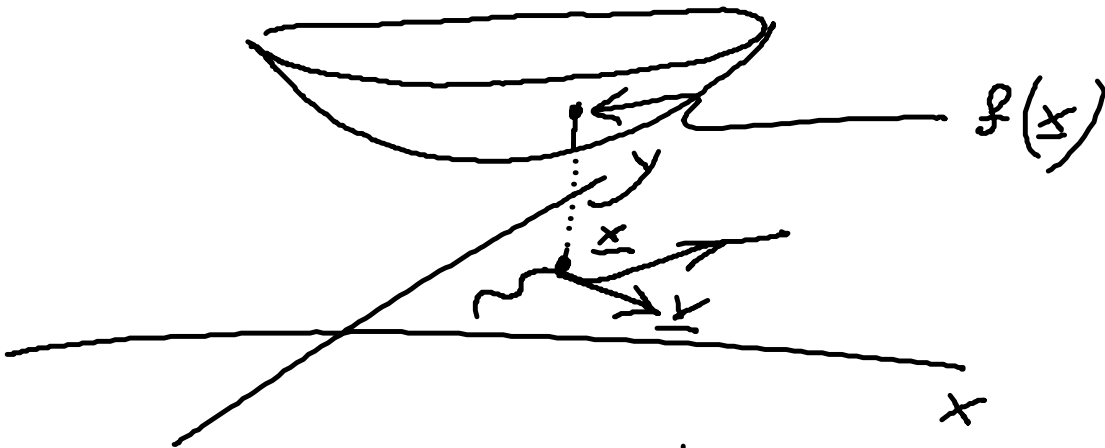
5.2. Gradient

Sei $f: \underline{x} \rightarrow f(\underline{x})$ ein Skalarfeld.

Dann heißt

$$D_{\underline{v}} f(\underline{x}) \equiv \frac{\partial}{\partial \underline{v}} f(\underline{x}) = \lim_{t \rightarrow 0} \frac{f(\underline{x} + t\underline{v}) - f(\underline{x})}{t}$$

Richtungsableitung von f in Richtung \underline{v}
an der Stelle \underline{x} .



$$\begin{aligned} \frac{\partial}{\partial \underline{v}} f(\underline{x}) &= \frac{d}{dt} f(\underline{x} + t\underline{v}) \Big|_{t=0} = \\ &= \frac{d}{dt} f(x_1 + tv_1, \dots, x_n + tv_n) \Big|_{t=0} \\ &= \sum_i \frac{\partial}{\partial x_i} f(\underline{x}) \underbrace{\frac{d}{dt} (x_i + tv_i)}_{v_i} \Big|_{t=0} = \frac{\partial f}{\partial x_i} v_i \\ &= \underline{v} \cdot \underline{\nabla} f(\underline{x}), \quad \text{mit} \end{aligned}$$

$$\underline{\nabla} f(x) \equiv \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right) \text{ - Gradient}$$

Richtungsableitung $\frac{\partial f}{\partial \underline{v}} = \left(\underline{\nabla} f(x), \underline{v} \right)$

- Der Gradient ist ein n -dim. Vektor
- Der Gradient gibt die Richtung des stärksten Anstiegs von f an.

Der Gradient in krummlinigen Koordinaten

$$\frac{d}{dt} f(\underline{x}(t)) = \frac{\partial f}{\partial x^i} \frac{dx^i}{dt} = \left(\underline{\nabla} f, \underline{v} \right), \quad \underline{v} = \dot{\underline{x}}$$

$$\frac{d}{dt} f(u^1(t), \dots, u^n(t)) = \frac{\partial f}{\partial u^i} \frac{du^i}{dt} \equiv \left(\underline{\nabla} f, \underline{v} \right)$$

geschw. $\underline{v} = \dot{u}^i \underline{g}_i$ in der lokalen Basis.

$$\left[\frac{\partial f}{\partial u^i} u^i = (\nabla f, \underline{v}) = \left(\nabla f, \underbrace{u^i g_i}_{\frac{\partial f}{\partial u^j} g^j} \right) = \frac{\partial f}{\partial u^j} \underbrace{g^j g_i}_{\delta^j_i} u^i = \frac{\partial f}{\partial u^i} u^i \right]$$

Also gilt $\boxed{\nabla f = \frac{\partial f}{\partial u^j} g^j}$ • kontravariante Basisvektoren.

Häufig ausdrücken durch die g_i
(kovariante Basis).

Häufig normiert $g_j^* \equiv \frac{g_j}{|g_j|}$

$$\nabla f = \frac{\partial f}{\partial u^j} g^j = \frac{\partial f}{\partial u^j} g^{ji} g_i = \frac{\partial f}{\partial u^j} g^{ji} |g_i| g_i^*$$

In orthogonalen Koordinaten gilt: $g_i g_j = g_{ij}$ ist diagonal.

dann $g_{ij} = \delta_{ij} |g_i|^2 \Rightarrow g^{ij} = \delta^{ij} \frac{1}{|g_i|^2}$

$$\Rightarrow \underline{\nabla} f = \frac{\partial f}{\partial u^j} \delta^{ij} |g_i| g_i^* \frac{1}{|g_i|^2}$$

$$= \frac{\partial f}{\partial u^i} \frac{g_i^*}{|g_i|}$$

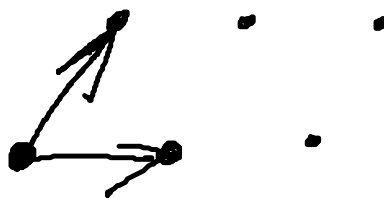
$$\underline{\nabla} f = \frac{\partial f}{\partial u^i} \frac{g_i^*}{|g_i|}$$

orthogonale KO,
normierte Basis
 g_i^*

Man braucht $|g_i| = \left| \frac{\partial \underline{r}}{\partial u^i} \right|$

Anwendung Festkörperphysik

Im \mathbb{R}^3 sei g_i eine (kovariante) Basis
eines fixen \mathcal{B} aus Punkten $\underline{x} = n^i g_i$



Bravais-
gitter.

Analyse mit Photonen, Elektronen etc.
mit ebenen Wellen

$$\psi_{\underline{k}} = e^{i \underline{k} \cdot \underline{r}}$$

Bedingung: $\Psi_{\underline{k}}(\underline{\Gamma}) = \Psi_{\underline{k}}(\underline{\Gamma} + \underline{x}), \quad \underline{x} \in \mathcal{B}$

Die Menge aller \underline{k} , die das erfüllt, heißt reziprokes Gitter.

reziprokes Gitter: Gitter im \underline{k} -Raum

$$e^{i\underline{k}(\underline{\Gamma} + \underline{x})} = e^{i\underline{k}\underline{\Gamma}}$$

$$e^{i\underline{k}\underline{x}} = 1 \Rightarrow \underline{k}\underline{x} = 2\pi m, \quad m \in \mathbb{Z}$$

Schreib $\underline{k} = 2\pi k_i g^i$ (Ansatz)

$$\Rightarrow \underbrace{\underline{k}\underline{x}}_{n^j g_j} = 2\pi k_i \underbrace{g^i n^j g_j}_{\delta^i_j} = 2\pi k_i n^i \rightarrow k_i \in \mathbb{Z}$$

Deshalb gilt $\underline{k} = 2\pi \cdot \text{ganze Zahl}_i \cdot g^i$

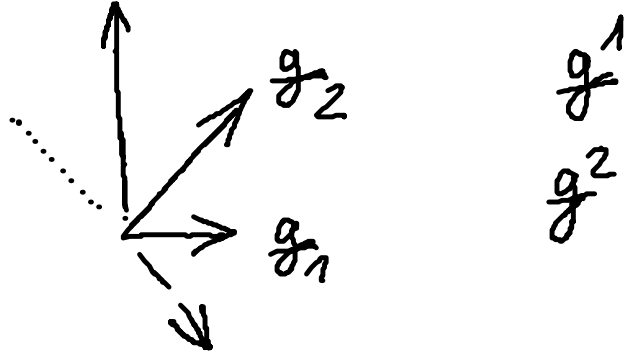
wobei $\underbrace{g^i}_{\text{Basis von } \mathcal{R}} \cdot \underbrace{g_j}_{\text{Basis von } \mathcal{B}} = \delta^i_j$
 (reziprokes Gitter)

Explicit :

$$g^1 = \frac{g_2 \times g_3}{g_1(g_2 \times g_3)}$$

$$g^2 = \frac{g_3 \times g_1}{g_1(g_2 \times g_3)}$$

$$g^3 = \frac{g_1 \times g_2}{\dots}$$



— \rightarrow