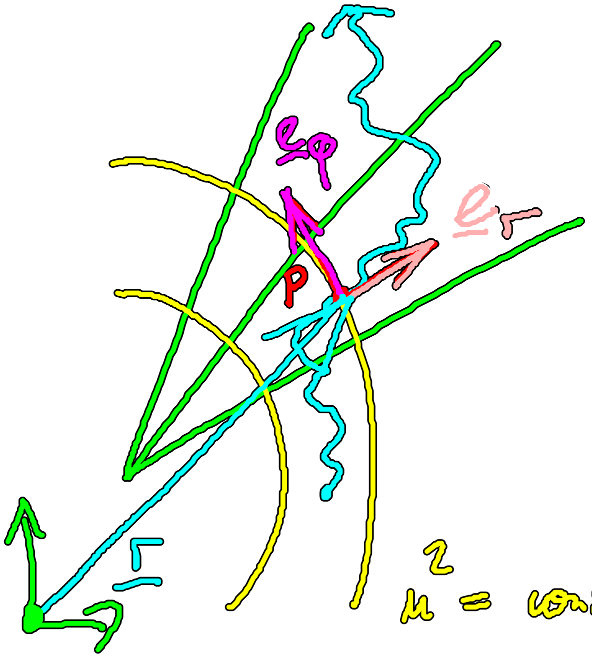
19.6.

19.6.



$u^1 = \text{const}$, $u^1 = \varphi$

$u^2 = \text{const}$

$u^2 = r$

$$\underline{g}_i \equiv \frac{\partial \underline{r}}{\partial u^i} = \sum_{k=1}^{d} \frac{\partial x^k}{\partial u^i} \underline{e}_k \equiv \frac{\partial x^k}{\partial u^i} \underline{e}_k$$

$$\underline{e}_k = \sum_j \frac{\partial u^j}{\partial x^k} \underline{g}_j \equiv \frac{\partial u^j}{\partial x^k} \underline{g}_j$$

Geschwindigkeit $\underline{v} = \dot{\underline{r}} = \frac{\partial x^k}{\partial u^i} \dot{u}^i \underline{e}_k$

$$= \dot{u}^i \underline{g}_i$$

Beschleunigung

$$\underline{a} = \dot{\underline{v}} = \frac{d}{dt} \dot{u}^i \underline{g}_i = \ddot{u}^i \underline{g}_i + \dot{u}^i \dot{\underline{g}}_i$$

$$\dot{\underline{g}}_i = \frac{d}{dt} \underline{g}_i(u^1, \dots, u^d) = \frac{\partial}{\partial u^j} \underline{g}_i \frac{d}{dt} u^j$$

$$= \underbrace{\frac{\partial}{\partial u^j} \underline{g}_i}_{\Gamma^l_{ij} \underline{g}_l} \dot{u}^j = \sum_{l,j} \Gamma^l_{ij} \underline{g}_l \dot{u}^j$$

Γ^l_{ij} heißen Christoffel-Symbole

Aus der Definition

$$\frac{\partial}{\partial u^j} g_i = \frac{\partial}{\partial u^j} \underbrace{\frac{\partial x^k}{\partial u^i} e_k}_{g_i} = \frac{\partial^2 x^k}{\partial u^j \partial u^i} e_k$$

$$e_k \rightsquigarrow \frac{\partial u^l}{\partial x^k} g^l$$

$$= \underbrace{\frac{\partial^2 x^k}{\partial u^j \partial u^i} \frac{\partial u^l}{\partial x^k}}_{\Gamma^l_{ij}} g^l$$

Also

$$\boxed{\Gamma^l_{ij} = \sum_{k=1}^{d} \frac{\partial^2 x^k}{\partial u^j \partial u^i} \frac{\partial u^l}{\partial x^k}}$$

Beschleunigung

$$\ddot{\underline{r}} = \ddot{u}^i g_i + \dot{u}^i \dot{g}_i$$

$$= \ddot{u}^i g_i + \dot{u}^i \Gamma^l_{ij} g_l \dot{u}^j$$

$$= \sum_l \left\{ \ddot{u}^l + \dot{u}^i \dot{u}^j \Gamma^l_{ij} \right\} g_l$$

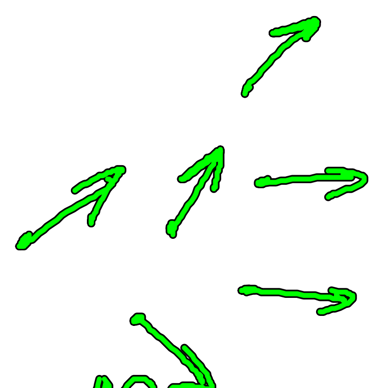
z.B. Newton $\underline{F} = m\,\underline{a}$ mit $\underline{F} = \sum_\ell f^\ell g_\ell$

$$\Rightarrow \quad \boxed{f^\ell = m\left(\ddot{u}^\ell + \dot{u}^i \dot{u}^j \Gamma^\ell_{ij}\right)}$$

s. PKO $\quad u^1 = r \quad u^2 = \varphi \quad u^3 = \theta \qquad \ell = 1,2,3$

<u>Literatur:</u> Klingbeil: „Tensorrechnung für Ingenieure"
Stephani: „Allgemeine Relativitätstheorie".

<u>V Vektoranalysis</u>



<u>Def:</u>

Abbildung $\mathbb{R}^n \to \mathbb{R}^m$, $\quad n,m \in \mathbb{N}$

$$\underline{x} = (x_1, \dots, x_n)^T \longmapsto \underline{f}(\underline{x}) = \big(f_1(x_1,\dots,x_n), \dots, f_m(x_1,\dots,x_n)\big)$$

heißt Vektorfeld.

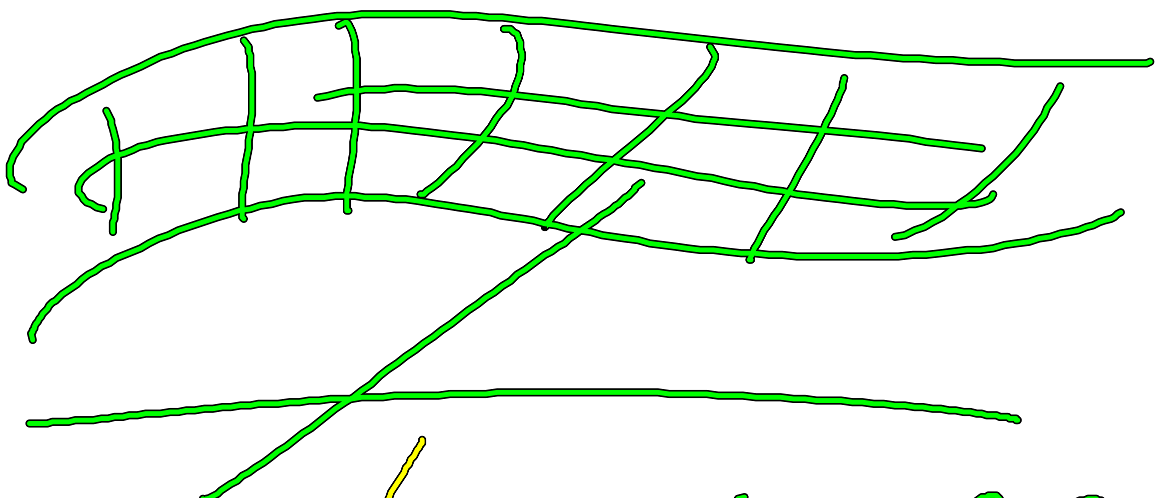
z.B. Elektrisches Feld $\underline{E}(\underline{x})$, $m=3$ $\quad \underline{E} = \vec{E} = \mathbf{E}$

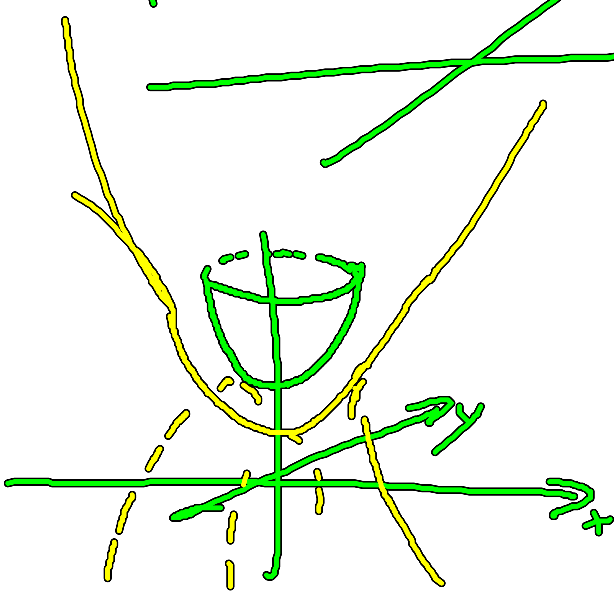
Temperatur $T(\underline{x})$, $m=1$ (fett gedruckt).

Bei $m=1$ spricht man von einem

<u>Skalarfeld</u>.

Skizze: Skalarfeld $f(x_1, \dots, x_n)$:





$$f(x,y) = x^2 + y^2 + 1$$

$$g(x,y) = x^2 - y^2$$

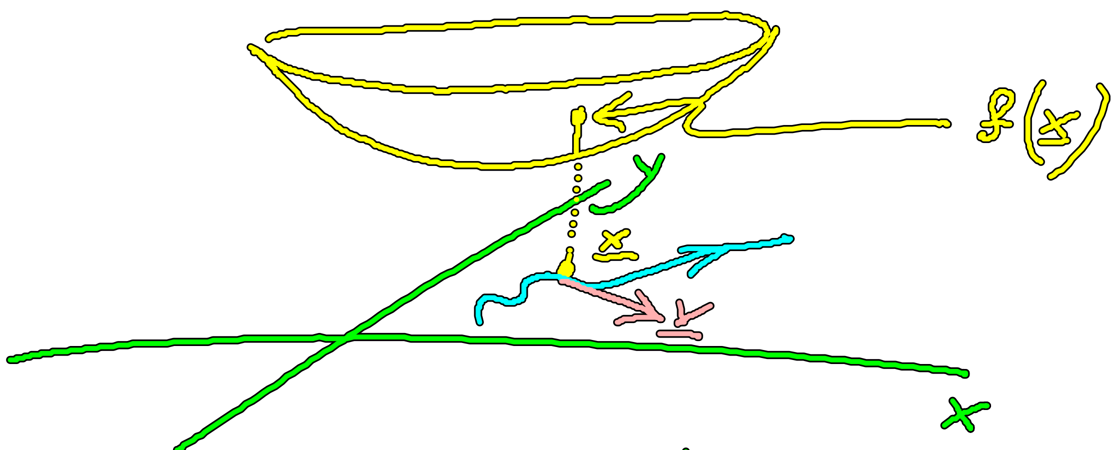
5.2. Gradient

Sei $f: \underline{x} \to f(\underline{x})$ ein Skalarfeld.

Dann heißt

$$D_{\underline{v}} f(\underline{x}) \equiv \frac{\partial}{\partial \underline{v}} f(\underline{x}) = \lim_{t \to 0} \frac{f(\underline{x}+t\underline{v}) - f(\underline{x})}{t}$$

Richtungsableitung von f in Richtung $\underline{v}$ an der Stelle $\underline{x}$.



$$\frac{\partial}{\partial \underline{v}} f(\underline{x}) = \frac{d}{dt} f(\underline{x}+t\underline{v})\Big|_{t=0} =$$

$$= \frac{d}{dt} f(x_1 + t v_1, \dots, x_n + t v_n)\Big|_{t=0}$$

$$= \sum_i \frac{\partial}{\partial x_i} f(\underline{x}) \underbrace{\frac{d}{dt}(x_i + t v_i)\Big|_{t=0}}_{v_i} = \frac{\partial f}{\partial x_i} v_i$$

$$= \underline{v} \cdot \underline{\nabla} f(\underline{x}), \quad \text{mit}$$

$$\underline{\nabla} f(x) \equiv \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right) \text{ - Gradient}$$

$$\boxed{\text{Richtungsableitung} \quad \frac{\partial f}{\partial \underline{v}} = \left(\underline{\nabla} f(\underline{x}), \underline{v}\right)}$$

- Der Gradient ist ein n-dim. Vektor
- Der Gradient gibt die Richtung des stärksten Anstiegs von f an.

Der Gradient in krummlinigen Koordinaten

$$\frac{d}{dt} f(\underline{x}(t)) = \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} = (\nabla f, \underline{v}), \quad \underline{v} = \dot{\underline{x}}$$

$$\frac{d}{dt} f\left(u^1(t), \dots, u^n(t)\right) = \frac{\partial f}{\partial u^i}\frac{du^i}{dt} \equiv (\nabla f, \underline{v})$$

Geschw. $\underline{v} = \dot{u}^i g_i$ in der lokalen Basis.

$$\frac{\partial f}{\partial u^i}\dot{u}^i = (\underline{\nabla} f, \underline{v}) = (\underbrace{\underline{\nabla} f}_{\frac{\partial f}{\partial u^j}\underline{g}^j}, \dot{u}^i \underline{g}_i) = \frac{\partial f}{\partial u^j}\underbrace{\underline{g}^j \underline{g}_i}_{\delta^j_i}\dot{u}^i$$

$$= \frac{\partial f}{\partial u^i}\dot{u}^i \quad ✓$$

Also gilt $\boxed{\underline{\nabla} f = \frac{\partial f}{\partial u^j}\underline{g}^j}$. kontravariante Basisvektoren.

Häufig auszudrücken durch die $\underline{g}_i$ (kovariante Basis).

Häufig normiert $\quad \underline{g}_j^* = \frac{\underline{g}_j}{|\underline{g}_j|}$

$$\underline{\nabla} f = \frac{\partial f}{\partial u^j}\underline{g}^j = \frac{\partial f}{\partial u^j} g^{ji}\underline{g}_i = \frac{\partial f}{\partial u^j} g^{ji}|\underline{g}_i|\underline{g}_i^*$$

In orthogonalen Koordinaten gilt: $\underline{g}_i \underline{g}_j = g_{ij}$ ist diagonal.

dann $g_{ij} = \delta_{ij}|\underline{g}_i|^2 \Rightarrow g^{ij} = \delta^{ij}\frac{1}{|\underline{g}_i|^2}$

$$\Rightarrow \quad \underline{\nabla} f = \frac{\partial f}{\partial u^j}\, \delta^{ij}\, |g_i|\, g_i^* \frac{1}{|g_i|^2}$$

$$= \frac{\partial f}{\partial u^i} \frac{g_i^*}{|g_i|}$$

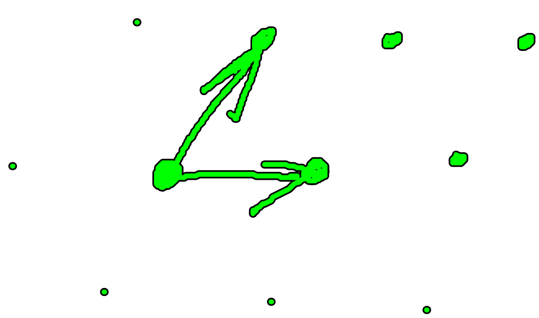
$$\boxed{\underline{\nabla} f = \frac{\partial f}{\partial u^i} \frac{g_i^*}{|g_i|}}$$

orthogonale KO, normierte Basis g_i^*

Man braucht $|g_i| = \left| \frac{\partial \underline{r}}{\partial u^i} \right|$

Anwendung Festkörperphysik

Im $\mathbb{R}^3$ sei g_i eine (kovariante) Basis eines Gitters $\mathcal{B}$ aus Punkten $\underline{x} = n^i g_i$



Bravais-Gitter.

Analyse mit Photonen, Elektronen etc.

mit ebenen Wellen

$$\psi_{\underline{k}} = e^{i \underline{k}\, \underline{r}}$$

Bedingung: $\Psi_{\underline{k}}(\underline{r}) = \Psi_{\underline{k}}(\underline{r}+\underline{x})$, $\underline{x} \in B$

Die Menge aller $\underline{k}$, die das erfüllt, heißt reziprokes Gitter.

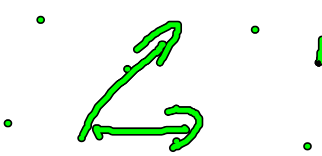
reziprokes Gitter: Gitter im **$\underline{k}$-Raum**

$$e^{i\underline{k}(\underline{r}+\underline{x})} = e^{i\underline{k}\underline{r}}$$

$$e^{i\underline{k}\underline{x}} = 1 \Rightarrow \underline{k}\cdot\underline{x} = 2\pi m, \quad m \in \mathbb{Z}$$

Schreibe $\underline{k} = 2\pi k_i \underline{g}^i$ (Ansatz)

$$\Rightarrow \underbrace{\underline{k}\,\underline{x}}_{m} = 2\pi k_i \underbrace{\underline{g}^i n^j \underline{g}_j}_{\delta^i_j} = 2\pi \underbrace{k_i n^i} \quad \to k_i \in \mathbb{Z}$$



$n^j \underline{g}_j$

Deshalb gilt $\underline{k} = 2\pi \cdot \text{ganze Zahl}_i \cdot \underline{g}^i$,

wobei $\underline{g}^i \cdot \underline{g}_j = \delta^i_j$

$\underline{g}^i$: Basis von R (reziprokes Gitter)

$\underline{g}_j$: Basis von B.

<u>Explizit</u> :

$$g^1 = \frac{g_2 \times g_3}{g_1 (g_2 \times g_3)}$$

$$g^2 = \frac{g_3 \times g_1}{g_1 (g_2 \times g_3)}$$

$$g^3 = \frac{g_1 \times g_2}{\cdots}$$