Superconductivity

- So far:
  - Infinite conductivity, perfect diamagnetism

    → London Equation $\underline{\nabla} \times \underline{j} + \frac{n_s e^{*2}}{m^* c} \underline{B} = 0$

    not just perfect conductivity!

    Derivable from extended boson state!

  - Length scales:

    $\lambda_L = \frac{m^* c^2}{4\pi n_s e^{*2}} = 41.9 \left(\frac{r_s}{a_0}\right)^{3/2} \left(\frac{n}{n_s}\right)^{1/2}$ London length

    $\xi(0)$ "Pippard coherence length" – nonlocal generalization of London

  - Flux quantization: $e^* = 2e$ – but not "simple" e^- pairs.

How to understand this?

1) Ginzburg – Landau theory

Inhomogeneous superconductor

$n_s \to n_s(\underline{r})$ volume density of s.c carriers condensate wave function $|\phi(\underline{r})|^2 = n_s(\underline{r})$

General choice : $\phi(\underline{r}) = \sqrt{n_s(\underline{r})} \cdot e^{i\Phi(\underline{r})}$

$$\underline{j}(\underline{r}) = \frac{ie^*}{2m^{**}} \left(\phi^* \underline{\nabla} \phi - \phi \underline{\nabla} \phi^* \right) - \frac{e^*}{2m^{**}} \underline{A}(\underline{r}) \, n_s(\underline{r})$$

Landau : - Phase transitions can be described by expanding F (free energy) around T_c

- where F is a functional of some order parameter

Use $\phi(\underline{r})$ as the order parameter

$$\begin{aligned} F_s &:= F - F_n \\ &= \int d^3r \Big[\alpha |\phi(\underline{r})|^2 + \frac{\beta}{2} |\phi(\underline{r})|^4 \\ &\qquad + \frac{1}{2m^{**}} \left| (\underline{\nabla} + ie\underline{A}(\underline{r}))\phi(\underline{r}) \right|^2 \\ &\qquad + \frac{B(\underline{r})^2}{8\pi} \Big] \end{aligned}$$

this is an expansion near T_c, i.e. $\frac{|T-T_c|}{T_c} \ll 1$

Approach: Minimize F as a function of ϕ

$\longrightarrow$ vary $\phi^*(\underline{r})$ by $\delta\phi^*(\underline{r})$ to get δF.

Results in :

$$\delta F = \int d^3r \, \delta\phi^*(\underline{r}) \Big[\alpha \phi(\underline{r}) + \beta |\phi(\underline{r})|^2 \phi(\underline{r})$$

$$- \frac{1}{2m^{**}} \left[(\underline{\nabla} + ie\, A(\underline{r}) \right]^2 \phi(\underline{r}) \Big]$$

$$+ \oint_{\text{surface}} d\underline{s}^2 \;\; \delta\phi^*(\underline{r}) \left[\underline{\nabla} + ie^* A(\underline{r}) \right] \phi(\underline{r})$$

↑
through integration by parts (Gauss' law) on $\underline{\nabla}\delta\phi^*$!

Equilibrium: $\delta F = 0$

→ both parts $\int d^3r \; \ldots$ and

$\oint d^2 s \; \ldots$

must vanish for finite $\delta\phi^*$!

$$\Longrightarrow \; - \frac{1}{2m^{**}} \left[\underline{\nabla} + ie^* \underline{A}(\underline{r}) \right]^2 \phi(\underline{r}) + \beta |\phi(\underline{r})|^2 \phi(\underline{r}) = - \alpha\, \phi(\underline{r}) \qquad (I)$$

and $\left[\underline{\nabla} + ie^* \underline{A}(\underline{r}) \right]_{\text{surface}} = 0 \qquad (II)$

(I) Ginzburg - Landau equation
(bulk superconductor)
two solutions for the hom. case $\phi(\underline{r}) \equiv \phi$ (const.)

1) $\phi(\underline{r}) = 0$

2) $|\phi(\underline{r})|^2 = n_s = -\frac{\alpha}{\beta}$

at $T \to T_c$: $n_s \to 0$

in lowest order: $\alpha \sim T - T_c$

$\beta = \text{const}\ (T)$

Use this for dimensionality considerations:

in 1D : $\phi(\underline{r}) \to n_s^{1/2} f(x)$, $\underline{A} = 0$

$$\Rightarrow \quad -\xi(T)^2 \frac{\partial^2 f}{\partial x^2} - f + f^3 = 0$$

$$\xi(T) = \frac{1}{\sqrt{2m^* |\alpha(T)|}}$$

is the "natural" length scale of the problem.

"coherence length" (Ginzburg-Landau)

"Ginzburg - Landau parameter":

$$\kappa = \frac{\lambda_L}{\xi}$$

NB: GL equ. + perturbation argument

$\Rightarrow$ London eq. like last time!

GL limitations: • $\left| \frac{T_c - T}{T} \right| \ll 1$

• General limitation:

fluctuations $|\delta \phi| \ll \sqrt{n_s}$

• can show $\left| \frac{T_c - T}{T} \right| \ll \kappa^2$

The critical field:

Consider boundary

if we assume infinitely sharp boundary: $\lambda_L = 0$, $\xi = 0$

can analyze this through Gibbs free energy equilibrium (applies for $\underline{H} = \text{const.}$)

$$G(T, N, H) = F(T, N, B) - \frac{1}{4\pi} \int d^3r \, \underline{H} \cdot \underline{B}(r)$$

$$g_s = f_s(T, N, 0) \qquad (\underline{B} = 0 \text{ in s.c.})$$

$$g_n = f_n(T, N, 0) + \frac{H^2}{8\pi} - \frac{H^2}{4\pi}$$

($H = B$ in the normal state $\mu \simeq 1$)

But H_c, critical field:

$g_s = g_n$ equilibrium

$$\Rightarrow f_s - f_n = -\frac{H_c^2}{8\pi} \stackrel{!}{=} \frac{\alpha^2}{2\beta}$$

$$\Rightarrow H_c = \sqrt{\frac{4\pi \, \alpha(0)^2}{\beta}} \cdot \frac{T_c - T}{T}$$

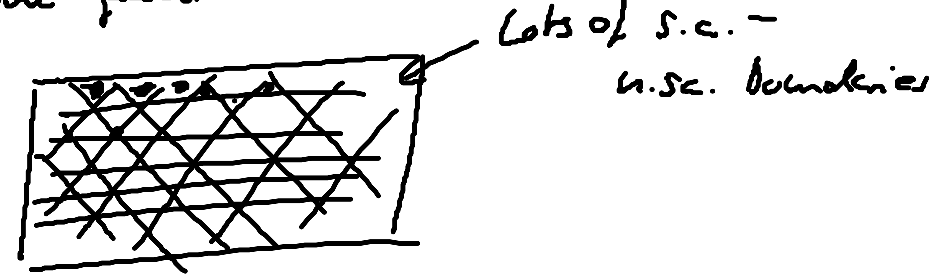
Can also do:

- Look at real surface energy ($\lambda_L, \xi \neq 0$)

  can show that surf. energy < 0 is possible

  if $\kappa > \frac{1}{\sqrt{2}}$.

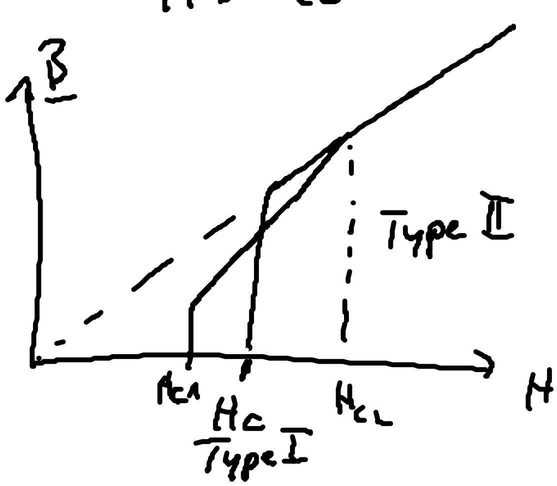
$\Longrightarrow$ boundaries between s.c., non-s.c. state become favorable ??



$\Rightarrow$ Type I vs. Type II superconductivity!

Type I: below a crit field $H_c(T)$
$\rightarrow$ no flux in the sample
above $H_c(T) \longrightarrow$ entire sample reverts to normal state!

Type II: $H < H_{c1}(T)$: No flux penetration
$H_{c1}(T) < H < H_{c2}(T)$: "mixed state"
$H > H_{c2}(T)$: revert to normal



(...)

But how about

2) microscopic theory?

Need: electron-(like) entities
must form pairs somehow.
- stable
- symmetric, $S=0$ or 1 (bosons)

<u>Cooper pairs</u>: Consider pairs formed in reciprocal space!

Fermi liquid: two quasiparticles $\underline{k}$, $-\underline{k}$
(time-reversal symmetric)

$\Rightarrow$ orbital wave function $\psi_0(\underline{r}_1, \underline{r}_2) = \sum_{\underline{k}} g_{\underline{k}} e^{i\underline{k}\underline{r}_1} e^{-i\underline{k}\underline{r}_2}$

Symmetry: Spin states
either $\underline{S}=0$ singlet $\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$
or $\underline{S}=1$ triplet $(\uparrow\uparrow), (\downarrow\downarrow), \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$

choose singlet:

$\Rightarrow \psi_0$ should be symmetric $\psi_0(\underline{r}_1, \underline{r}_2) = \sum_{k > k_F} g_{\underline{k}} \cos \underline{k}(\underline{r}_1 - \underline{r}_2)$

Schrödinger Eq.

$$H\psi_0 = \left[\sum_{i=1}^{2} \frac{p_i^2}{2m} + V(\underline{r}_1 - \underline{r}_2)\right]\psi_0 = E\psi_0$$

can reformulate in recip. space

$$(E - 2\epsilon_{\underline{k}})\, g_{\underline{k}} = \sum_{k' > k_F} V_{\underline{k},\underline{k}'}\, g_{\underline{k}'}$$

where $$V_{\underline{k},\underline{k}'} = \frac{1}{V} \int d^3r \; V(\underline{r})\, e^{i(\underline{k}' - \underline{k})\underline{r}}$$

But what is V ??

Free electrons: Coulomb potential:

$$V_{\underline{k},\underline{k}'} = V(\underbrace{\underline{k} - \underline{k}'}_{q}) = \frac{4\pi e^2}{q^2} > 0$$

But these are not free electrons!

Remember Thomas-Fermi: $\epsilon(k) = \frac{\underline{k}^2 + \underline{k}_0^2}{\underline{k}^2} \pm 1$

$$V \sim \frac{1}{r} \rightarrow \frac{1}{\epsilon\, r}$$

$$\rightarrow V(q) = \frac{4\pi e^2}{q^2 + k_0^2}$$ still positive

However: can show that ion cores (+) around can screen the e^-

$$V(q) \rightarrow \frac{4\pi e^2}{q + \underline{k}_0} \cdot \frac{\omega^2}{\omega^2 - \omega_q^2} \qquad \omega = \frac{1}{\hbar}(\epsilon_{\underline{k}} - \epsilon_{\underline{k}'})$$

Cooper: Phonons, Schmonons!

Let $V_{\underline{k},\underline{k}'} = -V$ up to $\left.\begin{matrix} |\epsilon_{\underline{k}} - \epsilon_F| \\ |\epsilon_{\underline{k}'} - \epsilon_F| \end{matrix}\right\} < \hbar\omega_D$

0 otherwise.

Then : $g_{\underline{k}} = V \frac{\sum_{\underline{k}'} g_{\underline{k}'}}{2\epsilon_{\underline{k}} - E}$ from above. $\Big| \sum_{\underline{k}}$ on both sides

$\Rightarrow \sum_{\underline{k}} g_{\underline{k}}$ cancels

$$\Rightarrow \frac{1}{V} = \sum_{k > k_F} (2\epsilon_{\underline{k}} - E)^{-1}$$

$$\int\limits_{\omega(\underline{k}) < \omega_D} d^3k \longrightarrow \int_{\epsilon_F}^{\epsilon_F + \hbar\omega_D} d\epsilon\, N(\epsilon) \frac{1}{2\epsilon - E} = \frac{1}{2} N(\epsilon_F) \ln \frac{2\epsilon_F - E + \hbar\omega_D}{2E_F - E}$$

"Weak coupling" : $N(\epsilon_F) \cdot V \ll 1$

Solve above eq. : $$E(\underbrace{e^{-\frac{2}{NV}} - 1}_{-1}) = 2\epsilon_F (e^{-\frac{2}{NV}} - 1) + 2\hbar\omega_D e^{-\frac{2}{NV}}$$

but

$$\Rightarrow E = 2\epsilon_F - 2\hbar\omega_D e^{-\frac{2}{N(0)V}} < 2\epsilon_F !$$

- Not analytic at $V = 0$ $\rightarrow$ No perturbation theory gets this result!
- binding by any negative V
- binding happens in $\underline{k}$ space, using time reversal symm. states.

How to <u>condense</u> Cooper pairs in a solid :

BCS theory (Bardeen, Cooper, Schrieffer)

Rough idea: could have g.s. $\Psi(\underline{r}_1, \dots, \underline{r}_N)$

$$= \phi(\underline{r}_1\sigma_1, \underline{r}_2\sigma_2) \cdot \dots$$

$$\cdot\, \phi(\underline{r}_{N-1}\sigma_{N-1}, \underline{r}_N\sigma_N)$$

$$\Psi_0 = \mathcal{a}\, \Psi \qquad \mathcal{a} \text{ antisymmetrization operator.}$$

How to do this in practice?

The fast route?

$$H = \sum_{\underline{k}\sigma} \xi_{\underline{k}}\, c^{+}_{\underline{k}\sigma}\, c_{\underline{k}\sigma} + \sum_{\underline{k}} c^{+}_{-\underline{k}\uparrow} c^{+}_{\underline{k}\downarrow} \sum_{\underline{k}'} V(\underline{k}, \underline{k}')\, c_{-\underline{k}'\downarrow} c_{\underline{k}'\uparrow}$$

$c^{+}_{\underline{k}\sigma}$, $c_{\underline{k}\sigma}$: creation, annihilation operators for e^- "quasiparticles" $\underline{k}\sigma$ from a ground state filled with e^- up to ϵ_F

e.g. $$c_k |\Psi\rangle = \pm n_{\underline{k}\sigma} |\tilde{\Psi}; n_{\underline{k}\sigma} - 1\rangle$$

$n_{\underline{k}\sigma} = 0, 1$ only!

$$c^{+}_{k} |\Psi\rangle = \pm (1 - n_{\underline{k}\sigma}) |\tilde{\Psi}; n_{\underline{k}\sigma} + 1\rangle$$

and $V(\underline{k}, \underline{k}')$ obviously couples Cooper pairs.

Mean-field theory for V term:

$$\sum_{\underline{k}'} V(\underline{k}, \underline{k}') \cdot (c_{-\underline{k}'\downarrow}\, c_{\underline{k}'\uparrow})$$

$$\approx \; -V \cdot \Theta(\omega_D - |\xi_{\underline{k}}|) \cdot \sum_{\underline{k}'} \Theta(\omega_D - |\xi_{\underline{k}'}|) \cdot \overline{(c_{-\underline{k}'\downarrow}\, c_{\underline{k}'\uparrow})}_{\text{average}}$$

$$=: \; -\Delta_{\underline{k}}$$

How to get this into BCS ??

rewrite $\quad c_{-\underline{k}\downarrow}\, c_{\underline{k}\uparrow} = \underbrace{b_{\underline{k}}}_{\langle c_{\underline{k}\downarrow} c_{\underline{k}\uparrow}\rangle_{\text{avg.}}} + (c_{-\underline{k}\downarrow}\, c_{\underline{k}\uparrow} - b_{\underline{k}})$

→ - throw this into BCS Ham.

- neglect $\quad (c_{-\underline{k}\downarrow}\, c_{\underline{k}\uparrow} - b_{\underline{k}})^2$ terms

$$H_{MF} = \sum_{\underline{k}\sigma} \xi_{\underline{k}}\, c^{*}_{-\underline{k}\sigma}\, c_{\underline{k}\sigma} - \sum_{\underline{k},\underline{k}'} V \left(c^{+}_{-\underline{k}\uparrow}\, c_{\underline{k}\downarrow}\, \underline{b}_{-\underline{k}'} + b^{+}_{\underline{k}}\, c_{-\underline{k}\downarrow}\, c_{\underline{k}\uparrow} - \underline{b}^{+}_{\underline{k}}\, \underline{b}_{\underline{k}'} \right)$$

$$= \ldots = \; -\text{"}- \; - \sum_{\underline{k}} \left(\Delta_{\underline{k}}\, c^{+}_{-\underline{k}\uparrow}\, c^{+}_{\underline{k}\downarrow} + \Delta^{+}_{\underline{k}}\, c_{-\underline{k}\downarrow}\, c_{\underline{k}\uparrow} - \Delta_{\underline{k}}\, b^{+}_{\underline{k}} \right)$$