

Superconductivity

• So far:

- Infinite conductivity, perfect diamagnetism

→ London Equation $\nabla \times \mathbf{j} + \frac{n_s e^* v}{m^* c} = 0$

not just perfect conductivity!

Derivable from extended boson state!

- Length scales:

$$\lambda_L = \frac{m^* c^2}{4\pi n_s e^{*2}} = 41.9 \left(\frac{m^*}{m_0}\right)^{3/2} \left(\frac{h}{n_s}\right)^{1/2} \text{ London length}$$

$\xi(0)$ "Pippard coherence length" -

non local generalization of London

- Flux quantization: $e^* = 2e$ - but not "simple" e^- pairs.

How to understand this?

1) Ginzburg-Landau theory

Inhomogeneous superconductor

$$n_s \rightarrow n_s(\mathbf{r})$$

volume density of s.c. carriers

condensate wave function $|\phi(\underline{r})|^2 = n_s(\underline{r})$

General choice: $\phi(\underline{r}) = \sqrt{n_s(\underline{r})} \cdot e^{i\Phi(\underline{r})}$

$$\underline{j}(\underline{r}) = \frac{i\hbar}{2m^*} (\phi^* \underline{\nabla} \phi - \phi \underline{\nabla} \phi^*) - \frac{e\hbar}{2m^*} \underline{A}(\underline{r}) n_s(\underline{r})$$

Landau: - Phase transitions can be described by expanding F (free energy) around T_c
- where F is a functional of some order parameter

Use $\phi(\underline{r})$ as the order parameter

$$F_S := F - F_n$$

$$= \int d^3r \left[\alpha |\phi(\underline{r})|^2 + \frac{\beta}{2} |\phi(\underline{r})|^4 + \frac{1}{2m^*} |(\underline{\nabla} + ie\underline{A}(\underline{r}))\phi(\underline{r})|^2 + \frac{B(\underline{r})^2}{8\pi} \right]$$

this is an expansion near T_c , i.e. $\frac{|T - T_c|}{T_c} \ll 1$

Approach: Minimize F as a function of ϕ

→ vary $\phi^*(\underline{r})$ by $\delta\phi^*(\underline{r})$ to get δF .

Results in:

$$\delta F = \int d^3r \delta\phi^*(\underline{r}) \left[\alpha \phi(\underline{r}) + \beta |\phi(\underline{r})|^2 \phi(\underline{r}) \right]$$

$$- \frac{1}{2m^*} [\nabla + ie A(\underline{r})]^2 \phi(\underline{r})]$$

$$+ \int_{\text{surface}} d\underline{S} \delta\phi^*(\underline{r}) [\nabla + ie^* A(\underline{r})] \phi(\underline{r})$$

↑
through integration by parts (Gauss' law) on $\nabla \delta\phi^*$!

Equilibrium: $\delta F = 0$

→ both parts $\int d^3r \dots$ and

$\int d\underline{S} \dots$
must vanish for finite $\delta\phi^*$!

$$\Rightarrow - \frac{1}{2m^*} [\nabla + ie^* A(\underline{r})]^2 \phi(\underline{r}) + \beta |\phi(\underline{r})|^2 \phi(\underline{r}) = -\alpha \phi(\underline{r}) \quad (\text{I})$$

and $[\nabla + ie^* A(\underline{r})]_{\text{surface}} = 0 \quad (\text{II})$

(I) Ginzburg - Landau equation
(bulk superconductor)

two solutions for the hom. case $\phi(\underline{r}) \equiv \phi(\text{const.})$

1) $\phi(\underline{r}) = 0$

2) $|\phi(\underline{r})|^2 = n_s = -\frac{\alpha}{\beta}$

at $T \rightarrow T_c$: $n_s \rightarrow 0$

in lowest order: $\alpha \sim T - T_c$
 $\beta = \text{const}(T)$

Use this for dimensionality considerations:

in 1D : $\phi(x) \rightarrow n_s^{1/2} f(x)$, $A=0$

$$\Rightarrow -\xi(T)^2 \frac{d^2 f}{dx^2} - f + f^3 = 0$$

$$\xi(T) = \frac{1}{\sqrt{2m^* |\alpha(T)|}}$$

is the "natural" length scale of the problem.

"coherence length" (Ginzburg-Landau)

"Ginzburg-Landau parameter":

$$\kappa = \frac{\lambda_L}{\xi}$$

NB: GL eqn. + perturbation against

\Rightarrow London eq. the last time!

GL limitations: • $\left| \frac{T_c - T}{T} \right| \ll 1$

• General limitation:

fluctuations $|\delta\phi| \ll \sqrt{n_s}$

• can show $\left| \frac{T_c - T}{T} \right| \ll \kappa^2$

The critical field:

Conductor boundary



if we assume

infinitely sharp

boundary: $\lambda_L = 0$, $f = 0$

can analyze this through Gibbs free energy equilibrium

(applies for $H = \text{const.}$)

$$G(T, N, H) = F(T, N, B) - \frac{1}{4\pi} \int d^3r \underline{H} \cdot \underline{B}(r)$$

$$g_s = f_s(T, N, 0) \quad (B=0 \text{ is s.c.})$$

$$g_n = f_n(T, N, 0) + \frac{H^2}{8\pi} - \frac{H^2}{4\pi}$$

($H=B$ is the normal state $\mu \approx 1$)

But H_c , critical field:

$$g_s = g_n \quad \text{equilibrium}$$

$$\Rightarrow f_s - f_n = -\frac{H_c^2}{8\pi} \stackrel{!}{=} \frac{\alpha^2}{2\beta}$$

$$\Rightarrow H_c = \sqrt{\frac{4\pi \alpha(0)^2}{\beta}} \cdot \frac{T_c - T}{T}$$

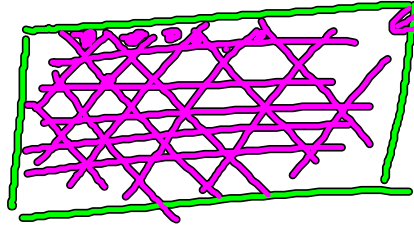
Can also do:

- Look at real surface energy ($\lambda_L, \xi \neq 0$)

can show that surf. energy < 0 is possible

if $\kappa > \frac{1}{\sqrt{2}}$.

→ boundaries between, s.c., non-s.c. state become favorable??



Lots of s.c. -
n.s.c. boundaries

⇒ Type I vs. Type II superconductivity!

Type I:

below a crit field $H_c(T)$

→ no flux in the sample

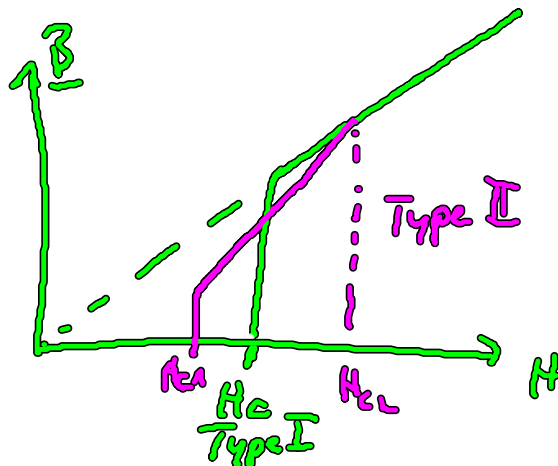
above $H_c(T)$ → entire sample reverts to normal state!

Type II:

$H < H_{c1}(T)$: No flux penetration

$H_{c1}(T) < H < H_{c2}(T)$: "mixed state"

$H > H_{c2}(T)$: revert to normal



(...)

But how about

2) microscopic theory?

Need: electron-(like) entities
must form pairs somehow.

- stable

- symmetric, $S=0$ or 1 (bosons)

Cooper pairs: Consider pairs formed in reciprocal space!

Fermi liquid: two quasiparticles $\underline{k}, -\underline{k}$
(time-reversal symmetric)

\Rightarrow orbital wave function $\Psi_0(\underline{r}_1, \underline{r}_2) = \sum_{\underline{k}} g_{\underline{k}} e^{i\underline{k}\underline{r}_1} e^{-i\underline{k}\underline{r}_2}$

Symmetry: Spin states

either $\underline{S}=0$

singlet $\frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow)$

or

$\underline{S}=1$

triplet $(\uparrow\uparrow), (\downarrow\downarrow), \frac{1}{\sqrt{2}} (\uparrow\downarrow + \downarrow\uparrow)$

choose singlet:

$\Rightarrow \Psi_0$ should be symmetric $\Psi_0(\underline{r}_1, \underline{r}_2) = \sum_{\underline{k} > k_F} g_{\underline{k}} \cos \underline{k} \cdot (\underline{r}_1 - \underline{r}_2)$

Schrödinger Eq.

$$H \Psi_0 = \left[\sum_{i=1}^2 \frac{p_i^2}{2m} + V(\underline{r}_1 - \underline{r}_2) \right] \Psi_0 = E \Psi_0$$

can reformulate in recip. space

$$(E - 2\epsilon_k) g_k = \sum_{k' > k_F} V_{k, k'} g_{k'}$$

$$\text{where } V_{k, k'} = \frac{1}{V} \int d^3r V(r) e^{i(k' - k)r}$$

But what is V ??

Free electrons: Coulomb potential:

$$V_{k, k'} = V(\underbrace{k - k'}_q) = \frac{4\pi e^2}{q^2} > 0$$

But these are not free electrons!

$$\text{Rieder Thomas - Fermi: } \epsilon(k) = \frac{k^2 + k_0^2}{k^2} \neq 1$$

$$V \sim \frac{1}{r} \rightarrow \frac{1}{e r}$$

$$\rightarrow V(q) = \frac{4\pi e^2}{q^2 + k_0^2} \quad \text{still positive}$$

However: can show that ion cores (+) around
can screen the e^-

$$V(q) \rightarrow \frac{4\pi e^2}{q + k_0} \cdot \frac{\omega^2}{\omega^2 - \omega_p^2} \quad \omega = \frac{1}{\hbar} (\epsilon_k - \epsilon_{k'})$$

Cooper: Phonons, Schuonons!

$$\text{Let } V_{k, k'} = -V \quad \text{up to } \begin{cases} |\epsilon_k - \epsilon_F| < \hbar\omega_p \\ |\epsilon_{k'} - \epsilon_F| < \hbar\omega_p \end{cases}$$

0 otherwise.

Then : $g_k = V \frac{\sum_{k'} g_{k'}}{2\epsilon_k - E}$ from above. | \sum_k on both sides

$$\Rightarrow \sum_k g_k \text{ cancels}$$

$$\Rightarrow \frac{1}{V} = \sum_{k > k_F} (2\epsilon_k - E)^{-1}$$

$$\int_{\omega(0) < \omega_0} d^3k \xrightarrow{\epsilon_F + \hbar\omega_0} \int_{\epsilon_F}^{\epsilon_F + \hbar\omega_0} d\epsilon N(\epsilon) \frac{1}{2\epsilon - E} = \frac{1}{2} N(\epsilon_F) \ln \frac{2\epsilon_F - E + \hbar\omega_0}{2\epsilon_F - E}$$

"weak coupling" : $N(\epsilon_F) \cdot V \ll 1$

Solve above eq. : $E \left(\underbrace{e^{-\frac{2}{N(V)}}}_{-1} - 1 \right) = 2\epsilon_F \left(e^{-\frac{2}{N(V)}} - 1 \right) + 2\hbar\omega_0 e^{-\frac{2}{N(V)}}$

but

$$\Rightarrow E = 2\epsilon_F - 2\hbar\omega_0 e^{-\frac{2}{N(V)}} < 2\epsilon_F!$$

- Not analytic at $V=0$ \rightarrow No perturbation theory gets this result!

- binding by any negative V

- binding happens in k space, using time-reversal symm. states.

How to condense Cooper pairs in a solid :

BCS theory (Bardeen, Cooper, Schrieffer)

Rough idea: could have g.s. $\Psi(\xi_1, \dots, \xi_N)$
 $= \phi(\xi_1 \sigma_1, \xi_2 \sigma_2) \cdot \dots$

$$\bar{\Psi}_0 = \alpha \Psi \quad \alpha \text{ antisymmetrization operator.}$$

$\cdot \phi(\xi_{N-1} \sigma_{N-1}, \xi_N \sigma_N)$

How to do this in practice?

The fast route?

$$H = \sum_{\underline{k}\sigma} \epsilon_{\underline{k}} c_{\underline{k}\sigma}^\dagger c_{\underline{k}\sigma} + \sum_{\underline{k}} c_{-\underline{k}\uparrow}^\dagger c_{\underline{k}\downarrow}^\dagger \sum_{\underline{k}'} V(\underline{k}, \underline{k}') c_{\underline{k}'\downarrow} c_{\underline{k}'\uparrow}$$

$c_{\underline{k}\sigma}^\dagger, c_{\underline{k}\sigma}$: creation, annihilation operators
 for e^- "quasiparticles" $\underline{k}\sigma$
 from a ground state filled with
 e^- up to E_F

e.g. $c_{\underline{k}} |\Psi\rangle = \pm n_{\underline{k}\sigma} |\tilde{\Psi}; n_{\underline{k}\sigma}-1\rangle$

$$n_{\underline{k}\sigma} = 0, 1 \text{ only!}$$

$$c_{\underline{k}}^\dagger |\Psi\rangle = \pm (1 - n_{\underline{k}\sigma}) |\tilde{\Psi}; n_{\underline{k}\sigma}+1\rangle$$

and $V(\underline{k}, \underline{k}')$ obviously couples Cooper pairs.

Mean-field theory for V term:

$$\sum_{\underline{k}'} V(\underline{k}, \underline{k}') \cdot (c_{-\underline{k}'\downarrow} c_{\underline{k}'\uparrow})$$

$$\approx -V \cdot \Theta(\omega_0 - |\xi_{\underline{k}}|)$$

$$\cdot \sum_{\underline{k}'} \Theta(\omega_0 - |\xi_{\underline{k}'}|) \cdot \overline{(c_{-\underline{k}} c_{\underline{k}'})_{\text{average}}}$$

$$=: -\Delta_{\underline{k}}$$

How to get this i-to BCS ??

rewrite $c_{-\underline{k}} c_{\underline{k}} = \underbrace{b_{\underline{k}} + (c_{-\underline{k}} c_{\underline{k}} - b_{\underline{k}})}_{\langle c_{-\underline{k}} c_{\underline{k}} \rangle_{\text{avg}}}$

→ throw this i-to BCS Ham.

- neglect $(c_{-\underline{k}} c_{\underline{k}} - b_{\underline{k}})^2$ terms

$$\begin{aligned} H_{MF} &= \sum_{\underline{k}\sigma} \xi_{\underline{k}} c_{-\underline{k}\sigma}^{\dagger} c_{\underline{k}\sigma} - \sum_{\underline{k}, \underline{k}'} V \left(c_{\underline{k}\uparrow}^{\dagger} c_{\underline{k}\downarrow} b_{-\underline{k}'} + b_{\underline{k}}^{\dagger} c_{\underline{k}\downarrow} c_{\underline{k}\uparrow} \right. \\ &\quad \left. - b_{\underline{k}}^{\dagger} b_{\underline{k}'} \right) \\ &= \dots = -^n - \sum_{\underline{k}} \left(\Delta_{\underline{k}} c_{-\underline{k}\uparrow}^{\dagger} c_{\underline{k}\downarrow}^{\dagger} + \Delta_{\underline{k}}^{\dagger} c_{-\underline{k}\downarrow} c_{\underline{k}\uparrow} - \Delta_{\underline{k}} b_{\underline{k}}^{\dagger} \right) \end{aligned}$$