6. Vektoranalysis

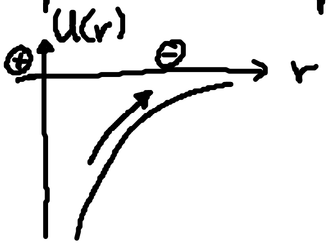
6.1 Skalarfelder

- Skalarfeld $f(\underline{r})$
- Bsp:

(1) ebene Welle: $f(\underline{r}) \sim \sin(\underline{k} \cdot \underline{r})$ (6.1)

(2) kugelsymmetrisches (Potential)feld: $U = U(r)$ (6.2)

Bsp: $U(r) \sim -\frac{1}{r}$ (6.3)



(3) zylindersymmetrisches (Potential)feld: $U = U(\varrho)$ (6.4)

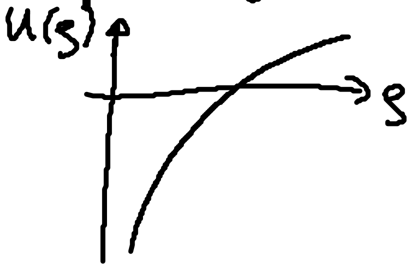
Zyl. Koord.
Abstand von der
z-Achse

„Äquipotential"flächen: $U(\varrho) = \text{konst}$

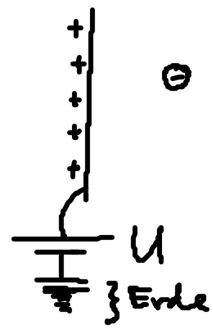
$\longrightarrow \varrho = \text{konst.}$

$\hat{=}$ Zylinderflächen um z-Achse

Bsp: $U(\rho) \sim \ln \rho$ (6.5)



... pot. Energie einer Testladung im Feld eines unendlich langen, homogen geladenen Drahtes



6.2 Vektorfelder

· Vektorfeld $\underline{a}(\underline{r})$: ordnet jedem Raumpkt. P einen Vektor $\in V_P$ zu

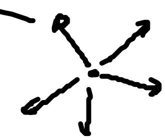
Bsp: Kraft $\underline{F}$, Geschwindigkeit (z.B. in Flüssigkeit)

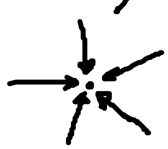
elektr. ($\underline{E}$)/magnet. ($\underline{H}$, $\underline{B}$,) Feld ...

· Bsp:

(1) Kugelsymmetr. (Quellen/Senken-) Feld:

$$\underline{a}(\underline{r}) = \pm \underbrace{a(r)}_{>0} \, \underline{e}_r \qquad (6.6)$$


„Quelle"


„Senke"

Bsp: $\underline{a}(\underline{r}) \sim \frac{1}{r^2} \, \underline{e}_r$ (6.7)

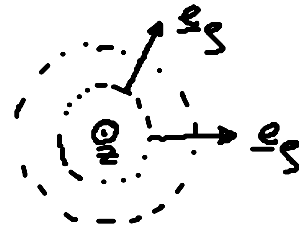
... (i) E-Feld einer Pkt.ladung

(ii) Kraft auf Test $\begin{cases} \text{masse} \\ \text{ladung} \end{cases}$

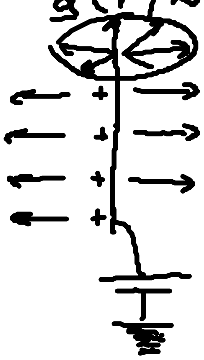
im Feld $\begin{cases} \text{eines Massenpktes} \\ \text{einer Pkt.ladung} \end{cases}$

(2) zylindersymmetr. Feld:

$$\underline{a}(\underline{r}) = a(\rho)\,\underline{e}_\rho \qquad (6.8)$$



Bsp: $\underline{a}(\underline{r}) \sim \frac{1}{\rho}\,\underline{e}_\rho$ (6.9)

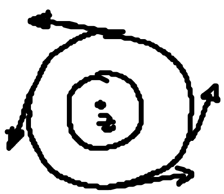


.... (i) E-Feld eines hom. gelad. Drahtes

(ii) Kraft auf Testladung im Feld eines hom. gelad. Drahtes

(3) (zylindersymmetr.) Wirbelfeld („Vortex")

$$\underline{v}(\underline{r}) = f(\rho, z)\,\omega\,\underline{e}_\varphi \qquad (6.10)$$



$|\underline{v}(r)| = f(\rho, z)\,\omega$

Flüssigkeiten, Tornados, Magnetfelder

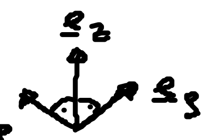
Bsp 1: $\underline{v}(\underline{r}) = \omega\rho\,\underline{e}_\varphi$ mit $|\underline{v}| = \omega\rho$!

$$\stackrel{!}{=} \underline{\omega} \times \underline{r}$$

„Wirbelstärke", hier: $\underline{\omega} = \omega\,\underline{e}_z$

Beweis:

$\underline{\omega} = \omega\,\underline{e}_z$, $\underline{r} = \rho\,\underline{e}_\rho + z\,\underline{e}_z$

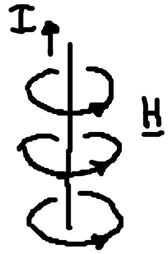


mit $\underline{e}_z \times \underline{e}_z = 0$

$\underline{e}_z \times \underline{e}_\rho = \underline{e}_\varphi$

$\longrightarrow \underline{\omega} \times \underline{r} = \omega\rho\,\underline{e}_\varphi$ qed

Bsp. 2: $\underline{H}(\underline{r}) \sim \frac{I}{\varrho}\,\underline{e}_\varphi$ (6.12)



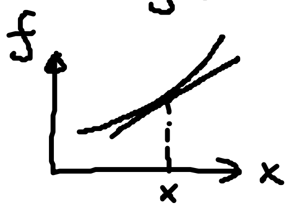
... Feld eines mit Strom I durchflossenen Leiters

6.3 Vollständiges Differential einer Funktion in 3D

• $\boxed{1D}$: Erinnerung: Geg: $f(x)$

Wert in Nachbarschaft von x: Taylorentwicklung!

$$f(x+dx) = \underbrace{f(x) + \frac{df}{dx}dx}_{\text{Gerade!}} + \underbrace{O(2)}_{\text{Terme }(dx)^n,\ n\geq 2} \quad (6.13)$$



Def:
$$\boxed{\begin{array}{l}\text{vollständiges Differential}\\ df(x) = \frac{df}{dx}dx\end{array}} \quad (6.14)$$

$$\xrightarrow{(6.13)} \quad f(x+dx) - f(x) = df + O(2) \quad (6.15)$$

• $\boxed{3D}$ Geg. $f(x_1, x_2, x_3)$

Wert in Nachbarschaft von $\{x_1, x_2, x_3\}$:

$$f(x_1+dx_1, x_2+dx_2, x_3+dx_3) \underset{\text{sukzessive}}{\overset{\text{mit (6.13)}}{=}}$$

$$= f(x_1+dx_1, x_2+dx_2, x_3) \quad + \quad \underbrace{\left.\frac{\partial f}{\partial x_3}\right|_{\substack{x_1+dx_1\\ x_2+dx_2\\ x_3}}}_{\left.\frac{\partial f}{\partial x_3}\right|_{\substack{x_1+dx_1\\ x_2,x_3}} + \underbrace{\frac{\partial^2 f}{\partial x_2 \partial x_3} dx_2}_{\longrightarrow O(2)}} dx_3$$

$$\vdots$$

$$f(x_1,x_2,x_3) + \left.\frac{\partial f}{\partial x_1}\right|_{x_1,x_2,x_3} dx_1 + \left.\frac{\partial f}{\partial x_2}\right|_{x_1,x_2,x_3} dx_2 \qquad \approx \left.\frac{\partial f}{\partial x_3}\right|_{x_1,x_2,x_3}$$

$$\longrightarrow f(x_1+dx_1, x_2+dx_2, x_3+dx_3) - f(x_1,x_2,x_3)$$

$$= df + O(2)$$

Def: $\boxed{\text{vollständiges Differential} \quad df = \frac{\partial f}{\partial x_i} dx_i}$ (6.17)

NB: $\frac{\partial f}{\partial x_i} = \left.\frac{\partial f}{\partial x_i}\right|_{x_j,\, j\neq i}$ halte $x_j \neq x_i$ fest

- Bsp: $f(r, \vartheta, \varphi) = r \sin\vartheta \cos\varphi$

$$\longrightarrow df = \sin\vartheta\cos\varphi\, dr + r\cos\vartheta\cos\varphi\, d\vartheta - r\sin\vartheta\sin\varphi\, d\varphi$$

- Vektorfeld $\underline{a}(x_1,x_2,x_3)$:

$$\boxed{d\underline{a} = \frac{\partial \underline{a}}{\partial x_i} dx_i} \qquad (6.18)$$

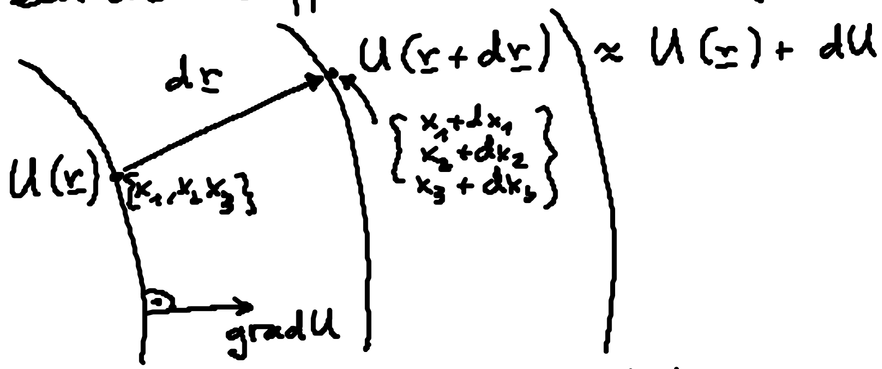
Bsp: kartesische Koord.:

$$d\underline{a} = \frac{\partial \underline{a}}{\partial x} dx + \frac{\partial \underline{a}}{\partial y} dy + \frac{\partial \underline{a}}{\partial z} dz \quad \text{mit} \quad \frac{\partial \underline{a}}{\partial x} = \frac{\partial a_x}{\partial x}\underline{e}_x + \frac{\partial a_y}{\partial x}\underline{e}_y + \frac{\partial a_z}{\partial x}\underline{e}_z$$

$$= \frac{\partial}{\partial x}\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

6.4. Der Nabla-Operator

- zentrale Größe der Vektoranalysis
- Führe ein über Differential eines Skalarfeldes $U(\underline{r})$:



Äquipotentialflächen $U(\underline{r})$

$$\boxed{d\underline{r} \overset{(6.18)}{=} \frac{\partial \underline{r}}{\partial x_i} dx_i \underset{(5.6)}{=} \left|\frac{\partial \underline{r}}{\partial x_i}\right| \underline{e}_i \, dx_i} \quad (6.19)$$

... Wegelement, „infinitesimaler" Differenzvektor

- einerseits:

$$dU(\underline{r}) = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 + \frac{\partial U}{\partial x_3} dx_3 \quad (6.20)$$

andererseits:

Def: Führe „Gradient von U" = grad U als Vektor ein, so daß: $dU(\underline{r}) = \text{grad}\, U \cdot d\underline{r}$ (6.21)

$$\longrightarrow \quad \boxed{\text{grad}\, U = \frac{1}{\left|\frac{\partial \underline{r}}{\partial x_i}\right|} \frac{\partial U}{\partial x_i} \underline{e}_i \quad \text{mit} \quad \underline{e}_i \cdot \underline{e}_j = \delta_{ij}} \quad (6.22)$$

... Gradientenfeld von U

Beweis: $dU \overset{?}{=} \text{grad}\, U \cdot d\underline{r} \underset{(6.19)}{\overset{(6.22)}{=}} \frac{1}{\left|\frac{\partial \underline{r}}{\partial x_i}\right|} \frac{\partial U}{\partial x_i} \underline{e}_i \cdot \left|\frac{\partial \underline{r}}{\partial x_j}\right| \underline{e}_j \, dx_j$

$$[\underline{e}_i \cdot \underline{e}_j = \delta_{ij}] = \frac{\partial U}{\partial x_i} dx_i \quad \text{qed}$$

• (6.22) legt nahe:

Def:
$$\boxed{\text{Nabla-Operator} \mathrel{\hat{=}} \text{Vektor-Differentialoperator} \\ \underline{\nabla} = \underline{e}_i \frac{1}{\left|\frac{\partial \underline{r}}{\partial x_i}\right|} \frac{\partial}{\partial x_i} \quad , \quad \text{so daß } \operatorname{grad} U = \underline{\nabla} U} \tag{6.23}$$

• entlang Äquipotentialflächen:

$$dU = 0 \xrightarrow{(6.21)} \operatorname{grad} U \perp d\underline{r}$$

$\longrightarrow \operatorname{grad} U \parallel$ Richtung maximaler Änderung von U

• Koordinatensysteme:

a) <u>kartesische Koordinaten</u>: $\left|\frac{\partial \underline{r}}{\partial x_i}\right| = 1\,!$

$$d\underline{r} = dx\,\underline{e}_x + dy\,\underline{e}_y + dz\,\underline{e}_z$$

$$(6.22) \longrightarrow \boxed{\begin{aligned} \operatorname{grad} U &= \underline{e}_x \frac{\partial U}{\partial x} + \underline{e}_y \frac{\partial U}{\partial y} + \underline{e}_z \frac{\partial U}{\partial z} \\ \underline{\nabla} &= \underline{e}_x \frac{\partial}{\partial x} + \underline{e}_y \frac{\partial}{\partial y} + \underline{e}_z \frac{\partial}{\partial z} \end{aligned}} \tag{6.24}$$

Bsp: $U \sim r^2 = x^2 + y^2 + z^2 \longrightarrow \underline{\nabla} U \sim 2x\,\underline{e}_x + 2y\,\underline{e}_y + 2z\,\underline{e}_z$
$= 2\underline{r}$

b) <u>Zylinderkoordinaten</u>

$$\left.\begin{aligned} d\underline{r} &\overset{(5.9)}{=} d\rho\,\underline{e}_\rho + \rho\, d\varphi\,\underline{e}_\varphi + dz\,\underline{e}_z \\ dU &= \frac{\partial U}{\partial \rho} d\rho + \frac{\partial U}{\partial \varphi} d\varphi + \frac{\partial U}{\partial z} dz \end{aligned}\right\} \quad \boxed{\underline{\nabla} = \underline{e}_\rho \frac{\partial}{\partial \rho} + \underline{e}_\varphi \frac{1}{\rho}\frac{\partial}{\partial \varphi} + \underline{e}_z \frac{\partial}{\partial z}}$$

[Dimensionsanalyse: $[\underline{\nabla}] = \frac{1}{\text{Länge}}$!!]

Bsp: $U(\rho) \sim \ln \rho \longrightarrow \underline{\nabla} U \sim \frac{1}{\rho}\,\underline{e}_\rho$

c) <u>Kugelkoordinaten</u>

$$\boxed{\underline{\nabla} = \underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\vartheta \frac{1}{r}\frac{\partial}{\partial \vartheta} + \underline{e}_\varphi \frac{1}{r \sin\vartheta}\frac{\partial}{\partial \varphi}} \tag{6.26}$$

Beweis: Übungen

- <u>Rechenregeln:</u> (Beweis: Übungen)

(i) $\underline{\nabla}(cU) = c\,\underline{\nabla}U \quad , \; c \in \mathbb{R}$

$\underline{\nabla}(U+V) = \underline{\nabla}U + \underline{\nabla}V$ (6.27)

$\underline{\nabla}(UV) = (\underline{\nabla}U)V + U(\underline{\nabla}V)$

(ii) $\underline{\nabla}(\underline{a}\cdot\underline{r}) = \underline{a} \qquad \underline{a}$ .. konstanter Vektor

$\underline{\nabla} r = \hat{\underline{r}} \quad , \quad \underline{r} = r\,\hat{\underline{r}} = r\,\underline{e}_r$ (6.28)

$\underline{\nabla} f(r) = \frac{\partial f}{\partial r}\,\hat{\underline{r}}$

insbesondere: $\underline{\nabla}\frac{1}{r} = -\frac{\hat{\underline{r}}}{r^2}$

- Richtungsableitung: „Ableitung entlang $\hat{\underline{v}}$" , $|\hat{\underline{v}}| = 1$

Def: $\boxed{\text{Richtungsableitung } \hat{\underline{v}}\cdot\underline{\nabla}U \text{ so daß mit } d\underline{r} = \hat{\underline{v}}\,ds: \quad dU = (\hat{\underline{v}}\cdot\underline{\nabla}U)\,ds}$ (6.29)