

6. Vektoranalysis

6.1 Skalarfelder

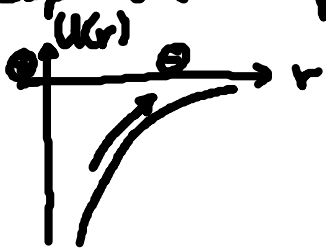
• Skalarfeld $f(x)$

• Bsp:

(1) ebene Welle: $f(x) \sim \sin(\frac{t}{c} \cdot x)$ (6.1)

(2) kugelsymmetrisches (Potential)feld: $U = U(r)$ (6.2)

Bsp: $U(r) \sim -\frac{1}{r}$ (6.3)



(3) zylindersymmetrisches (Potential)feld: $U = U(\rho)$ (6.4)

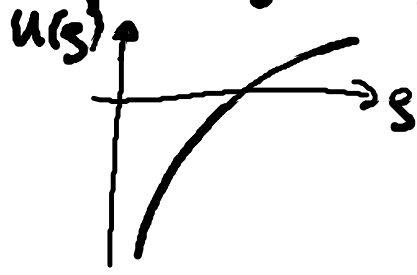
↑
Zyl. Koord.
Abstand von der
z-Achse

„Äquipotential“ flächen: $U(\rho) = \text{const}$

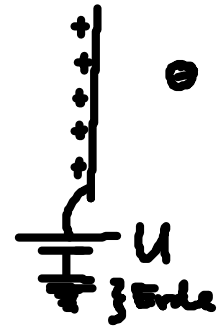
→ $\rho = \text{const.}$

∴ Zylinderflächen um z-Achse

Bsp: $U(s) \sim \ln s$ (6.5)



... pot. Energie einer Testladung im Feld eines unendlich langen, homogen geladenen Drahtes

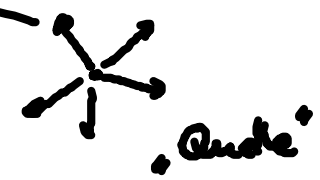
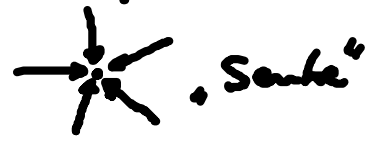


6.2 Vektorfelder

- Vektorfeld $\underline{g}(x)$: ordnet jedem Raumpkt. P einen Vektor $\in V_p$ zu
 Bsp: Kraft \underline{E} , Geschwindigkeit (z.B. in Flüssigkeit)
 elektr. (\underline{E}) / magnet. ($\underline{H}, \underline{K}$) Feld...

Bsp: (1) Kugel symmetr. (Quellen/Senken-) Feld:

$$\underline{g}(r) = \pm \underbrace{a(r)}_{>0} \underline{e}_r \quad (6.6)$$



Bsp: $\underline{g}(r) \sim \frac{1}{r^2} \underline{e}_r$ (6.7)

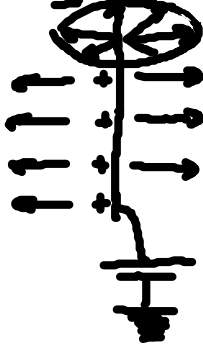
- ... (i) E-Feld einer Pkt. Ladung
- (ii) Kraft auf Test $\left\{ \begin{array}{l} \text{masse} \\ \text{Ladung} \end{array} \right.$
 im Feld $\left\{ \begin{array}{l} \text{eines Massenpunktes} \\ \text{einer Pkt. Ladung} \end{array} \right.$

(2) zylindrisch symmetr. Feld:

$$\underline{a}(\underline{r}) = a(\rho) \underline{e}_\rho \quad (6.8)$$



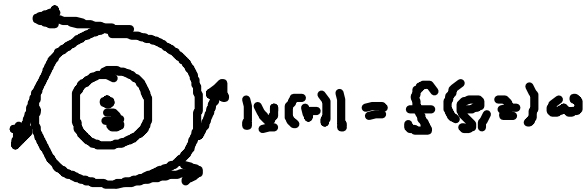
Bsp: $\underline{a}(\underline{r}) \sim \frac{1}{s} \underline{e}_s \quad (6.9)$



- (i) E-Feld eines hom. gelad. Drahtes
-
- (ii) Kraft auf Testladung im Feld eines hom. gelad. Drahtes

(3) (zylindrisch symmetr.) Wirbelfeld („Vortex“)

$$\underline{v}(\underline{r}) = f(\rho, z) \omega \underline{e}_\phi \quad (6.10)$$



Flüssigkeiten, Tornados, Magnetfelder

Bsp 1: $\underline{v}(\underline{r}) = \omega \rho \underline{e}_\phi$ mit $|\underline{v}| = \omega \rho$!

$$\stackrel{!}{=} \underline{\omega} \times \underline{r}$$

„Wirbelstärke“, hier: $\underline{\omega} = \omega \underline{e}_z$

Beweis:

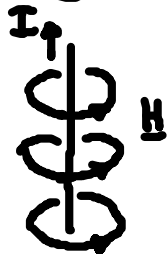
$$\underline{\omega} = \omega \underline{e}_z, \quad \underline{r} = \rho \underline{e}_\rho + z \underline{e}_z \quad \underline{e}_\rho, \underline{e}_\phi, \underline{e}_z$$

$$\text{mit } \underline{e}_z \times \underline{e}_z = 0$$

$$\underline{e}_z \times \underline{e}_\rho = \underline{e}_\phi$$

$$\rightarrow \underline{\omega} \times \underline{r} = \omega \rho \underline{e}_\phi \quad \text{qed}$$

Bsp. 2: $H(r) \sim \frac{I}{r} e_\varphi$ (6.12)



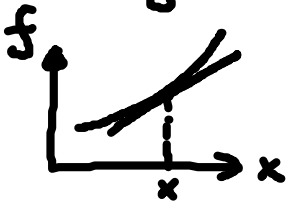
... Feld eines mit Strom I durchflossenen Leiters

6.3 Vollständiges Differential einer Funktion in 3D

• [10] Erinnerung: Geg: $f(x)$

Wert in Nachbarschaft von x : Taylorentwicklung!

$$f(x+dx) = f(x) + \underbrace{\frac{df}{dx} dx}_{\text{Gerade!}} + \underbrace{O(2)}_{\text{Parabel } (dx)^n, n \geq 2} \quad (6.13)$$



Def:

vollständiges Differential

$$df(x) = \frac{df}{dx} dx \quad (6.14)$$

$$\stackrel{(6.13)}{\longrightarrow} f(x+dx) - f(x) = df + O(2) \quad (6.15)$$

• [30] Geg: $f(x_1, x_2, x_3)$

Wert in Nachbarschaft von $\{x_1, x_2, x_3\}$:

$$f(x_1+dx_1, x_2+dx_2, x_3+dx_3) \stackrel{\text{mit (6.13)}}{\approx} \text{additive}$$

$$\begin{aligned}
&= f(x_1+dx_1, x_2+dx_2, x_3) + \underbrace{\frac{\partial f}{\partial x_3} \Big|_{\substack{x_1+dx_1 \\ x_2+dx_2 \\ x_3}}}_{x_3} dx_3 \\
&\quad \vdots \\
&f(x_1, x_2, x_3) + \frac{\partial f}{\partial x_1} \Big|_{x_1, x_2, x_3} dx_1 + \frac{\partial f}{\partial x_2} \Big|_{x_1, x_2, x_3} dx_2 + \frac{\partial^2 f}{\partial x_2 \partial x_3} dx_2 dx_3 \rightarrow O(2) \\
&\quad + \frac{\partial f}{\partial x_2} \Big|_{x_1, x_2, x_3} dx_2 \approx \frac{\partial f}{\partial x_3} \Big|_{x_1, x_2, x_3} dx_3
\end{aligned}$$

$$\begin{aligned}
\longrightarrow f(x_1+dx_1, x_2+dx_2, x_3+dx_3) - f(x_1, x_2, x_3) \\
= df + O(2)
\end{aligned}$$

Def: vollständiges Differential (6.17)

$$df = \frac{\partial f}{\partial x_i} dx_i$$

NB: $\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_i} \Big|_{x_j, j \neq i}$ ↖ halte $x_j = x_j$ fest

• Bsp: $f(r, \vartheta, \varphi) = r \sin \vartheta \cos \varphi$

$$\longrightarrow df = \sin \vartheta \cos \varphi dr + r \cos \vartheta \cos \varphi d\vartheta - r \sin \vartheta \sin \varphi d\varphi$$

• Vektorfeld $\underline{g}(x_1, x_2, x_3)$:

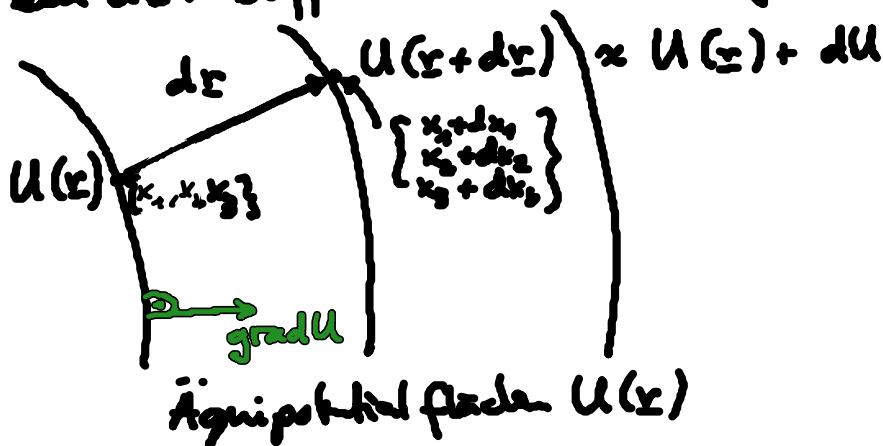
$$d\underline{g} = \frac{\partial \underline{g}}{\partial x_i} dx_i \quad (6.18)$$

Bsp: kartesische Koord.:

$$\begin{aligned}
d\underline{g} &= \frac{\partial \underline{g}}{\partial x} dx + \frac{\partial \underline{g}}{\partial y} dy + \frac{\partial \underline{g}}{\partial z} dz \quad \text{mit} \quad \frac{\partial \underline{g}}{\partial x} = \frac{\partial \underline{g}}{\partial x} \underline{e}_x + \frac{\partial \underline{g}}{\partial x} \underline{e}_y + \frac{\partial \underline{g}}{\partial x} \underline{e}_z \\
&= \frac{\partial \underline{g}}{\partial x} \begin{pmatrix} \underline{e}_x \\ \underline{e}_y \\ \underline{e}_z \end{pmatrix}
\end{aligned}$$

6.4. Der Nabla-Operator

- zentrale Größe der Vektoranalysis
- Führe ein über Differential eines Skalarfeldes $U(\underline{x})$:



$$d\underline{x} \stackrel{(6.1)}{=} \frac{\partial \underline{x}}{\partial x_i} dx_i \stackrel{(5.6)}{=} \left| \frac{\partial \underline{x}}{\partial x_i} \right| \underline{e}_i dx_i \quad (6.13)$$

... Wegelament, „infinitesimaler“ Differenzvektor

• einerseits:

$$dU(\underline{x}) = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 + \frac{\partial U}{\partial x_3} dx_3 \quad (6.20)$$

• andererseits:

Def: Führe „Gradient von U “ = $\text{grad } U$ als Vektor ein, so daß: $dU(\underline{x}) = \text{grad } U \cdot d\underline{x}$ (6.21)

→ $\text{grad } U = \frac{1}{\left| \frac{\partial \underline{x}}{\partial x_i} \right|} \frac{\partial U}{\partial x_i} \underline{e}_i$ mit $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$ (6.22)

... Gradientenfeld von U

Beweis: $dU \stackrel{?}{=} \text{grad } U \cdot d\underline{x} \stackrel{(6.22)}{=} \frac{1}{\left| \frac{\partial \underline{x}}{\partial x_i} \right|} \frac{\partial U}{\partial x_i} \underline{e}_i \cdot \left| \frac{\partial \underline{x}}{\partial x_j} \right| \underline{e}_j dx_j$ (6.13)

$$[\underline{e}_i \cdot \underline{e}_j = \delta_{ij}] = \frac{\partial U}{\partial x_i} dx_i \quad \text{qed}$$

• (6.22) legt nahe:

Def: Nabla-Operator $\hat{=}$ Vektor-Differentialoperator
$$\underline{\nabla} = \mathbf{e}_i \left| \frac{\partial \mathbf{x}}{\partial x_i} \right| \frac{\partial}{\partial x_i}, \quad \text{so dass } \text{grad} U = \underline{\nabla} U \quad (6.23)$$

• entlang Äquipotentialflächen:

$$dU = 0 \xrightarrow{(6.21)} \text{grad } U \perp d\mathbf{x}$$

\rightarrow grad U \parallel Richtung maximale Änderung von U

• Koordinatensysteme:

a) Kartesische Koordinaten: $\left| \frac{\partial \mathbf{x}}{\partial x_i} \right| = 1!$

$$d\mathbf{x} = dx \mathbf{e}_x + dy \mathbf{e}_y + dz \mathbf{e}_z$$

$$(6.22) \rightarrow \text{grad } U = \mathbf{e}_x \frac{\partial U}{\partial x} + \mathbf{e}_y \frac{\partial U}{\partial y} + \mathbf{e}_z \frac{\partial U}{\partial z} \quad (6.24)$$

$$\underline{\nabla} = \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z}$$

Bsp: $U \sim r^2 = x^2 + y^2 + z^2 \rightarrow \underline{\nabla} U \sim 2x \mathbf{e}_x + 2y \mathbf{e}_y + 2z \mathbf{e}_z = 2\mathbf{x}$

b) Zylinderkoordinaten

$$d\mathbf{x} \stackrel{(53)}{=} ds \mathbf{e}_s + s d\varphi \mathbf{e}_\varphi + dz \mathbf{e}_z$$

$$dU = \frac{\partial U}{\partial s} ds + \frac{\partial U}{\partial \varphi} d\varphi + \frac{\partial U}{\partial z} dz$$

$$\underline{\nabla} = \mathbf{e}_s \frac{\partial}{\partial s} + \mathbf{e}_\varphi \frac{1}{s} \frac{\partial}{\partial \varphi} + \mathbf{e}_z \frac{\partial}{\partial z}$$

[Dimensionsanalyse:
 $[\underline{\nabla}] = \frac{1}{\text{Länge}} \parallel$]

Bsp: $U(s) \sim \ln s \rightarrow \underline{\nabla} U \sim \frac{1}{s} \mathbf{e}_s$

c) Kugelkoordinaten

$$\underline{\nabla} = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \quad (6.25)$$

Beweis: Übungen

• Rechenregeln: (Beweis: Übungen)

$$\begin{aligned} \text{(i)} \quad \underline{\nabla}(cU) &= c \underline{\nabla}U, \quad c \in \mathbb{R} \\ \underline{\nabla}(U+V) &= \underline{\nabla}U + \underline{\nabla}V \\ \underline{\nabla}(UV) &= (\underline{\nabla}U)V + U(\underline{\nabla}V) \end{aligned} \quad (6.27)$$

$$\begin{aligned} \text{(ii)} \quad \underline{\nabla}(\underline{a} \cdot \underline{r}) &= \underline{a} \quad \underline{a} \text{ - konstanter Vektor} \\ \underline{\nabla}r &= \hat{\underline{r}}, \quad \underline{r} = r \hat{\underline{r}} = r \underline{e}_r \\ \underline{\nabla}f(r) &= \frac{\partial f}{\partial r} \hat{\underline{r}} \\ \text{insbesondere: } \underline{\nabla} \frac{1}{r} &= -\frac{\hat{\underline{r}}}{r^2} \end{aligned} \quad (6.28)$$

• Richtungsableitung: „Ableitung entlang $\hat{\underline{\varphi}}$ “, $|\hat{\underline{\varphi}}| = 1$

Def: Richtungsableitung $\hat{\underline{\varphi}} \cdot \underline{\nabla}U$
so daß mit $d\underline{r} = \hat{\underline{\varphi}} ds$: $dU = (\hat{\underline{\varphi}} \cdot \underline{\nabla}U) ds$ (6.29)