6.4 Der Nabla-Operator

- $$\boxed{\underline{\nabla} = \underline{e}_i \frac{1}{\left|\frac{\partial \underline{r}}{\partial x_i}\right|} \frac{\partial}{\partial x_i}, \quad \text{so daß } \operatorname{grad} U = \underline{\nabla} U} \quad (6.23)$$

- vollständiges Differential: $dU = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 + \frac{\partial U}{\partial x_3} dx_3$

$$\boxed{dU(\underline{r}) = \operatorname{grad} U \cdot d\underline{r}} \quad (6.21)$$

$$dU = 0 \longrightarrow \operatorname{grad} U \perp d\underline{r}$$

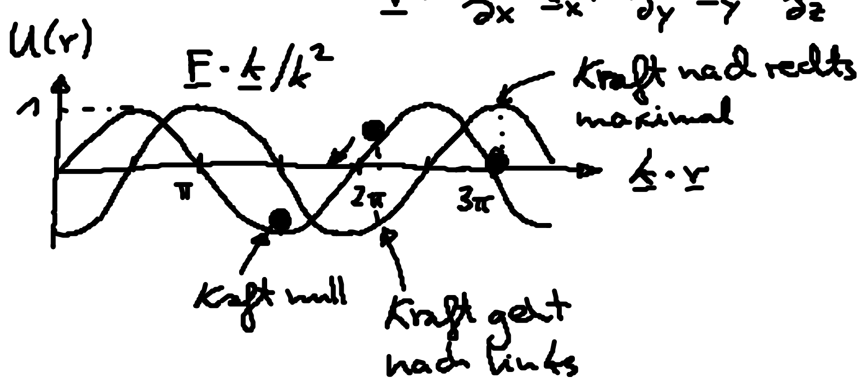
- $\boxed{\begin{array}{l}\text{Richtungsableitung } \hat{\underline{v}} \cdot \underline{\nabla} U \\ \text{so daß mit } d\underline{r} = \hat{\underline{v}}\, ds: \quad dU = (\hat{\underline{v}} \cdot \underline{\nabla} U)\, ds\end{array}}$ (6.29)

- Beispiel: $U(\underline{r})$ ... potentielle Energie
  $\longrightarrow \underline{F}(\underline{r}) = -\underline{\nabla} U(\underline{r})$ ... Kraftfeld
  } s. Mechanik (6.30)

(i) $U(\underline{r}) \sim \sin(\underline{k} \cdot \underline{r}) \xrightarrow[\text{kartes. Koord.}]{} \underline{F}(\underline{r}) \sim -\underline{k} \cos(\underline{k} \cdot \underline{r})$ (6.31)

$$\underline{k} \cdot \underline{r} = k_x x + k_y y + k_z z$$

$$\underline{\nabla} = \frac{\partial}{\partial x} \underline{e}_x + \frac{\partial}{\partial y} \underline{e}_y + \frac{\partial}{\partial z} \underline{e}_z$$



} Bewegung in Potentiallandschaft

(ii) $U(r) \sim \frac{1}{r}$ ... kugelsymmetrisches / Zentral-Potential

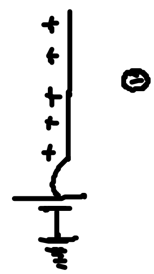
$$\xrightarrow[\text{Kugelkoord}]{(6.28)} \quad \underline{F}(\underline{r}) \sim \frac{\underline{e}_r}{r^2} \qquad (6.32)$$

... Kraft ∥ radialer Richtung

... Gravitationskraft im Schwerefeld eines Planeten (vgl. Kap. 6.1 & 6.2)

(iii) $U(\rho) \sim \ln \rho$ ... zylindersymmetr. Potential

$$\xrightarrow[\text{zyl. Koord.}]{(6.25)} \quad \underline{F}(\underline{r}) \sim -\frac{1}{\rho}\, \underline{e}_\rho \qquad (6.33)$$ (vgl. Kap. 6.1 & 6.2)



6.5 Divergenz

- Erinnerung: $\text{grad}\, U(\underline{r}) = \underline{\nabla}\, U(\underline{r})$ ... Vektor

  $\longrightarrow$ weitere Operationen von $\underline{\nabla}$?

- Def.: Divergenz eines Vektorfeldes $\underline{a}(\underline{r})$

$$\boxed{\text{div}\, \underline{a}(\underline{r}) = \underline{\nabla} \cdot \underline{a}(\underline{r})} \qquad (6.34)$$

  ... Quellenfeld von $\underline{a}(\underline{r})$

- Kartesische Koord.:

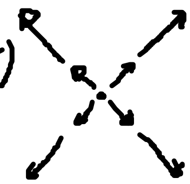
$$\underline{\nabla} \overset{(6.24)}{=} \underline{e}_i \frac{\partial}{\partial x_i}, \quad \underline{a}(\underline{r}) = a_i(\underline{r})\, \underline{e}_i, \quad i = x, y, z$$

$$\longrightarrow \quad \underline{\nabla}\cdot\underline{a} = \left(\underline{e}_i \frac{\partial}{\partial x_i}\right)\cdot\left(a_j(\underline{r})\,\underline{e}_j\right)$$

$$\left[\frac{\partial}{\partial x_i}\underline{e}_j = 0\right] = \left(\frac{\partial}{\partial x_i} a_j\right)\underbrace{\underline{e}_i\cdot\underline{e}_j}_{\delta_{ij}}$$

$$\longrightarrow \quad \boxed{\underline{\nabla}\cdot\underline{a}(\underline{r}) = \frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z} \qquad (6.35)$$

• Bsp 1: $\underline{a}(\underline{r}) = \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (6.36)



$$\longrightarrow \quad \underline{\nabla}\cdot\underline{r} = 1+1+1 = 3$$

Bsp 2: Kugelsymmetr. Quellen-/Senkenfeld:

$$\underline{a}(\underline{r}) = a(r)\,\underline{e}_r = \frac{a(r)}{r}\,\underline{r} \quad , \quad \underline{e}_r = \frac{\underline{r}}{r} \qquad (6.37)$$

$$\underline{\nabla}\cdot\underline{a}(\underline{r}) = \frac{a(r)}{r}\underbrace{\underline{\nabla}\cdot\underline{r}}_{3} + \frac{\partial}{\partial r}a(r)/r \; \underbrace{\underline{\nabla}\cdot(r\,\underline{r})}_{\frac{\partial}{\partial x_i}(r)\,x_i} = \underline{e}_r\cdot\underline{r} = r$$

$$= 2\,\frac{a(r)}{r} + \frac{\partial a}{\partial r} \qquad (6.38)$$

für $a(r) = \frac{1}{r^2}$ [s. Kap. 6.2]

$$\underline{\nabla}\cdot\underline{a}(\underline{r}) = 0\,! \qquad (6.39)$$

für $r \neq 0$ (Singularität)

also: Punktmasse/-ladung bei $r=0$ erzeugt Feld, sonst keine Quellen!

• **Deutung**: $\boxed{\text{$\underline{\nabla}\cdot\underline{a}(\underline{r})$ identifiziert lokal Quellen \& Senken von Vektorfeldern = „Flüssen"}}$ (6.40)

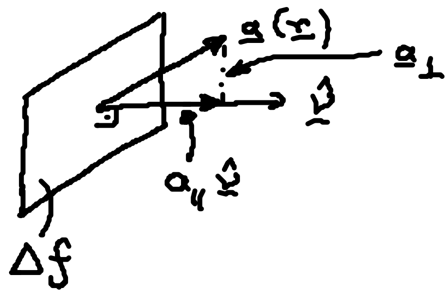
Bsp: $\underline{a}(\underline{r}) = \underline{v}(\underline{r})$ ... Geschw. feld einer Flüssigkeit

aber auch: $\underline{E}(\underline{r})$, nicht $\underline{B}(\underline{r})$, da div $\underline{B}(\underline{r}) = 0$

„keine magnetischen Monopole"

Motivation:

(1) Fluß durch Fläche mit Normale $\underline{\hat{\nu}}$ $(|\underline{\hat{\nu}}|=1)$ und Größe Δf



$a_\parallel \Delta f = (\underline{a}\cdot\underline{\hat{\nu}})\Delta f$ tritt durch Fläche

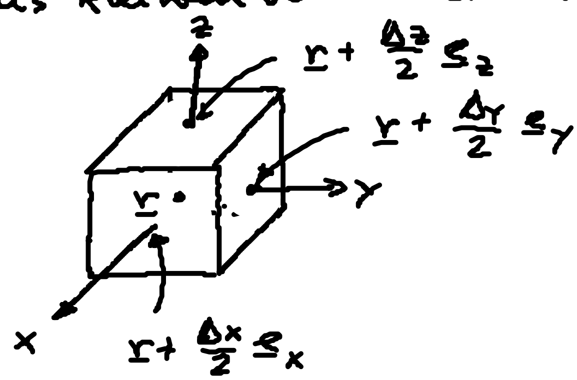
$\underline{a}_\perp \perp \underline{\hat{\nu}}$ nicht!

insbesondere: $\underline{\hat{\nu}} = \pm \underline{e}_x \longrightarrow a_\parallel = \pm a_x$

$\phantom{insbesondere: \underline{\hat{\nu}}} = \pm \underline{e}_y \longrightarrow \ldots$

$\vdots$

(2) Fluß aus kleinem Volumenelement $\Delta V = \Delta x\, \Delta y\, \Delta z$ um $\underline{r}$



Aufgabe: Summe aller Flüsse aus ΔV hinaus!

Konvention: $\boxed{\underline{\hat{\nu}} \text{ zeigt } \underline{\text{immer}} \text{ aus } \Delta V \text{ hinaus}}$ (6.41)

Grund: $\underline{a}\cdot\underline{\hat{\nu}} > 0$ für Quelle!

hier:
$$q_x^\pm = \overset{\underline{\hat{\nu}} = \underline{e}_x}{\underset{\underline{\hat{\nu}} = -\underline{e}_x}{\pm}} a_x\left(\underline{r} \pm \frac{\Delta x}{2}\underline{e}_x\right) \overset{\text{Taylor}}{=} \pm\left[a_x(\underline{r}) \pm \frac{\Delta x}{2}\frac{\partial}{\partial x}a_x\right]$$

$$q_y^\pm = \pm a_y\left(\underline{r} \pm \frac{\Delta y}{2}\underline{e}_y\right) = \pm\left[a_y(\underline{r}) \pm \frac{\Delta y}{2}\frac{\partial}{\partial y}a_y\right]$$

$$q_z^\pm = \ldots\ldots = \pm\left[a_z(\underline{r}) \pm \frac{\Delta z}{2}\frac{\partial}{\partial z}a_z\right]$$

$\longrightarrow$ Fluß aus ΔV:

$$q(\underline{r})\,\Delta V = (q_x^+ + q_x^-)\,\Delta y\,\Delta z$$
$$+ (q_y^+ + q_y^-)\,\Delta x\,\Delta z$$
$$+ (\;\ldots\;)$$
$$= \left\{\left[\cancel{a_x(\underline{r})} + \frac{\Delta x}{2}\frac{\partial}{\partial x}a_x\right] - \left[\cancel{a_x(\underline{r})} - \frac{\Delta x}{2}\frac{\partial}{\partial x}a_x\right]\right\}\Delta y\,\Delta z$$
$$+ \;\ldots\; \Delta x\,\Delta z \;+\; \ldots\; \Delta x\,\Delta y$$
$$= \underbrace{\left(\frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z\right)}_{\mathrm{div}\,\underline{a}}\;\underbrace{\Delta x\,\Delta y\,\Delta z}_{\Delta V}$$

$$\boxed{\begin{array}{l} q(\underline{r})\,\Delta V \quad \text{mit } q(\underline{r}) = \mathrm{div}\,\underline{a}(\underline{r}) = \underline{\nabla}\cdot\underline{a}(\underline{r}) \\ \text{mißt Fluß aus } \Delta V \text{ heraus} \end{array}} \quad (6.43)$$

Fälle: (i) $q(\underline{r}) = 0$ ... „was rein fließt, fließt auch raus"

(ii) >0 ... „es fließt mehr raus als rein" $\hat{=}$ Quellen

(iii) <0 .. Senke

Bsp: $\underline{v}(\underline{r}) = v_0\,\underline{r} \longrightarrow \mathrm{div}\,\underline{v} = 3\,v_0$ (6.44)

$\longrightarrow$ überall Quellen!

· Regeln:
$$\left.\begin{array}{ll} (1) & \underline{\nabla}\cdot(\underline{a}+\underline{b}) = \underline{\nabla}\cdot\underline{a} + \underline{\nabla}\cdot\underline{b} \\ (2) & \underline{\nabla}\cdot[f(\underline{r})\,\underline{a}] = f(\underline{r})\,\underline{\nabla}\cdot\underline{a} + \underline{a}\cdot\underline{\nabla}f(\underline{r}) \end{array}\right\} \quad (6.45)$$

Beweis: in kartesischen Koord.!

· <u>Zylinderkoordinaten:</u>

$$\underline{\nabla}\cdot\underline{a}(\underline{r}) = \left(\underline{e}_\rho\frac{\partial}{\partial\rho} + \underline{e}_\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \underline{e}_z\frac{\partial}{\partial z}\right)\cdot\left(a_\rho\,\underline{e}_\rho + a_\varphi\,\underline{e}_\varphi + a_z\,\underline{e}_z\right)$$

Achtung:

$$\left.\begin{aligned}
&(i) \quad \frac{\partial}{\partial \varphi} \underline{e}_\rho = \frac{\partial}{\partial \varphi}\begin{pmatrix}\cos\varphi \\ \sin\varphi \\ 0\end{pmatrix} = \begin{pmatrix}-\sin\varphi \\ \cos\varphi \\ 0\end{pmatrix} = \underline{e}_\varphi \\
&\qquad\qquad\qquad\qquad\qquad \longrightarrow \frac{1}{\rho} a_\rho \\
&(ii) \quad \frac{\partial}{\partial \varphi} \underline{e}_\varphi = -\underline{e}_\rho \,, \quad \text{aber } \underline{e}_\varphi \cdot \underline{e}_\rho = 0\,! \\
&(iii) \quad \frac{\partial}{\partial x_i} \underline{e}_j = 0 \quad \text{sonst}
\end{aligned}\right\} \quad (6.46)$$

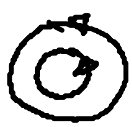
(6.46) und $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$

$$\longrightarrow \boxed{\begin{aligned}\underline{\nabla} \cdot \underline{a}(\underline{r}) &= \frac{\partial}{\partial \rho} a_\rho + \frac{1}{\rho} a_\rho + \frac{1}{\rho}\frac{\partial}{\partial \varphi} a_\varphi + \frac{\partial}{\partial z} a_z \\ &= \frac{1}{\rho}\frac{\partial(\rho a_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial}{\partial \varphi} a_\varphi + \frac{\partial}{\partial z} a_z\end{aligned}} \quad (6.47)$$

Bsp: (1) $\underline{E}(\underline{r}) \sim -\frac{1}{\rho}\underline{e}_\rho$ ... Feld eines geladenen Drahtes

$$\longrightarrow \operatorname{div} \underline{E}(\underline{r}) \sim \frac{1}{\rho^2} - \frac{1}{\rho^2} = 0 \,, \quad \rho \neq 0$$

(2) $\underline{v}(\underline{r}) = \omega \rho\, \underline{e}_\varphi \longrightarrow \operatorname{div} \underline{v} = \frac{1}{\rho}\frac{\partial}{\partial \varphi}(\omega \rho) = 0$



- **Kugelkoordinaten:** $\quad \underline{a} = a_r \underline{e}_r + a_\vartheta \underline{e}_\vartheta + a_\varphi \underline{e}_\varphi$

$$\boxed{\begin{aligned}\underline{\nabla} \cdot \underline{a}(\underline{r}) &= \frac{\partial a_r}{\partial r} + \frac{2}{r} a_r + \frac{1}{r}\frac{\partial a_\vartheta}{\partial \vartheta} + \frac{1}{r\tan\vartheta} a_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial a_\varphi}{\partial \varphi} \\ &= \frac{1}{r^2}\frac{\partial(r^2 a_r)}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial(\sin\vartheta\, a_\vartheta)}{\partial \vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial a_\varphi}{\partial \varphi}\end{aligned}} \quad (6.48)$$

Beweis: Übungen!

Bsp: $\underline{a}(\underline{r}) = \underline{r} = r\,\underline{e}_r \longrightarrow \operatorname{div} \underline{a} = \frac{\partial r}{\partial r} + \frac{2}{r} r = 3\,!$

[vgl. (6.36)]