4. Vektoren und Koordinatensysteme

4.1 Kinematik

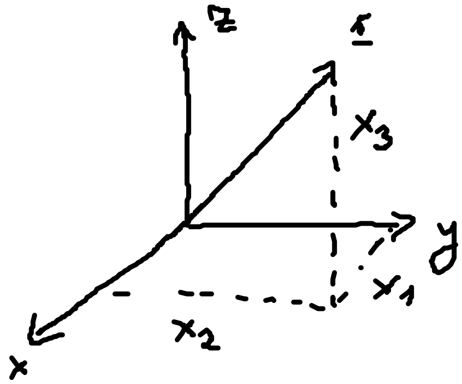
Bewegung eines Massenpunktes charakterisiert durch

Ortsvektor $\underline{r}(t) \in \mathbb{R}^3$

Geschwindigkeitsvektor $\underline{v}(t) := \dot{\underline{r}}(t) = \frac{d}{dt}\underline{r}(t)$

Beschleunigungsvektor $\underline{a}(t) := \ddot{\underline{r}}(t)$

Kartesisches Koordinatensystem



Rechtssystem!

$$\underline{r} \,\hat{=}\, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (x_1, x_2, x_3)^T$$

Komponenten

- Skalarprodukt (inneres Produkt): $\underline{a} \cdot \underline{b} = \sum_{i=1}^{3} a_i b_i$

  Notation auch $(\underline{a}, \underline{b})$, $(\vec{a}, \vec{b})$, $\underline{a}\,\underline{b}$

- Vektorprodukt (äußere Produkt, Kreuzprodukt)

$$\underline{a} \times \underline{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

  Einheitsvektoren $\underline{e}_1, \underline{e}_2, \underline{e}_3$

- dyadische Produkt (Tensorprodukt)

  $a \otimes b$, auch $\underline{a}\,\underline{b}$

  Tensor 2. Stufe mit Komponenten $a_i b_j$

$$\underline{a} \otimes \underline{b} = \begin{pmatrix} a_1 b_1 & \cdots & a_n b_1 \\ a_1 b_2 & & \vdots \\ \vdots & & \\ a_1 b_n & \cdots & a_n b_n \end{pmatrix}$$

Aufgabenstellung

Berechne die Bahnkurve $\underline{r}(t)$ aus vorgegebener Beschleunigung $\underline{a}(t)$

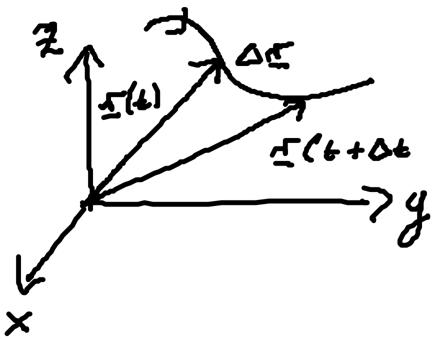
Anfangsbed.: $\underline{r}(0) = \underline{r}_0$

$\underline{v}(0) = \underline{v}_0$

Lösung $\underline{v}(t) = \underline{v}_0 + \int_0^t \underline{a}(t')\,dt'$

$\underline{r}(t) = \underline{r}_0 + \underline{v}_0 t + \int_0^t \left[\int_0^{t'} a(t'')\,dt'' \right] dt'$

a) Kartesische Koordinaten



$\underline{r}(t) = \sum_{j=1}^{3} x_j(t)\,\underline{e}_j$

Komponenten

kartes. Basisvektoren

$|\underline{e}_j| = 1$

zeitunabhängig Einheitsvektoren

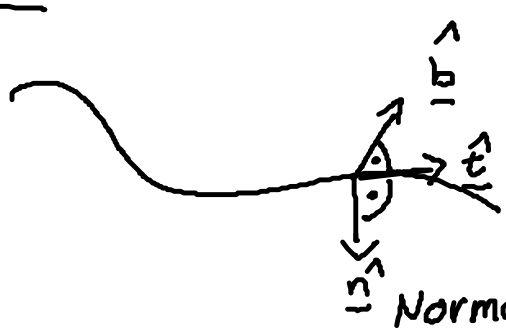
Geschwindigkeitsvektor $\underline{v}(t) = \sum_{j=1}^{3} \dot{x}_j(t)\,\underline{e}_j = \lim_{\Delta t \to 0} \frac{\underline{r}(t+\Delta t) - r(t)}{\Delta t}$

tangential zur Bahnkurve

Beschleunigung $\underline{a}(t) = \sum_{j=1}^{3} \ddot{x}_j(t)\,\underline{e}_j$

b) Natürliche Koordinaten

Begleitendes Dreibein

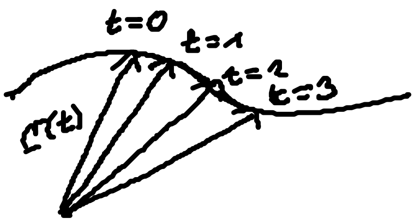


Das begleitende Dreibein (orthonormiertes Rechtssystem) ist in jedem Punkt der Bahnkurve

<u>Tangenteneinheitsvektor</u> $\quad \hat{\underline{t}} := \dfrac{\frac{d\underline{r}}{dt}}{\left|\frac{d\underline{r}}{dt}\right|}$

Parametrisierung der Bahnkurve $\underline{r}(t)$ nach der <u>Bogenlänge</u> $s(t)$

$$s(t) := \int_0^t \left|\frac{d\underline{r}(t')}{dt}\right| dt = \lim_{\Delta t \to 0} \sum_{n=0}^{N-1} \left|\underline{r}(t_{n+1}) - \underline{r}(t_n)\right|$$



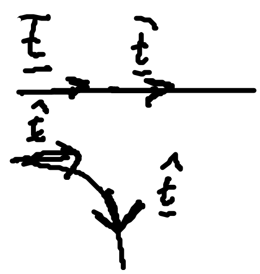
Also: $\dfrac{ds}{dt} = \left|\dfrac{d\underline{r}}{dt}\right|$

$$\Rightarrow \quad \hat{\underline{t}} = \frac{\frac{d\underline{r}}{dt}}{\frac{ds}{dt}} = \frac{d\underline{r}}{dt} \cdot \frac{dt}{ds} = \frac{d\underline{r}(s)}{ds}$$

($\hat{\underline{t}}$ liegt also tangential zur Bahnkurve in Richtung wachsender Bogenlänge)

(Neue Parametrisierung $\tilde{\underline{r}}(s) = \underline{r}(t(s))$ statt $\underline{r}(t)$)
$\sim$ dann weglassen

Änderung der Richtung von $\hat{\underline{t}}(s)$



$K = 0$

$K > 0$

Maß für die <u>Krümmung</u> $\quad K := \left|\dfrac{d\hat{\underline{t}}(s)}{ds}\right|$

<u>Krümmungsradius</u> $\quad \tilde{\rho} := \dfrac{1}{K}$

Normaleneinheitsvektor $\quad \hat{\underline{n}} := \dfrac{\frac{d\hat{\underline{t}}}{ds}}{\left|\frac{d\hat{\underline{t}}}{ds}\right|} = \dfrac{1}{K} \dfrac{d\hat{\underline{t}}}{ds}$

$\hat{\underline{n}} \perp \hat{\underline{t}}$

(Beweis: $\quad \hat{\underline{t}} \cdot \hat{\underline{t}} = 1 \quad \Rightarrow 0 = \dfrac{d}{ds}(\hat{\underline{t}} \cdot \hat{\underline{t}}) = 2\hat{\underline{t}}\dfrac{d\hat{\underline{t}}}{ds} = 0$)

$\hat{\underline{n}}$ und $\hat{\underline{t}}$ spannen die <u>Schmiegungsebene</u> auf (momentane Bahnebene)

Binormalen einheitsvektor $\underline{\hat{b}} := \underline{\hat{t}} \times \underline{\hat{n}}$

$\underline{\hat{b}} \perp \underline{\hat{t}}, \underline{\hat{n}}$ (d.h. $\underline{\hat{b}}$ = const, falls Bewegung in fester Ebene)

Geschwindigkeitsvektor: $\underline{v}(t) = \frac{d\underline{r}}{dt} = \underbrace{\frac{d\underline{r}}{ds}}_{\underline{\hat{t}}} \cdot \frac{ds}{dt} \Rightarrow |\underline{v}| = \frac{ds}{dt}$

$$\boxed{\underline{v}(t) = v\,\underline{\hat{t}}}$$

Betrag

Richtung

Beschleunigungsvektor

$$\underline{a}(t) = \frac{d}{dt}(v\,\underline{\hat{t}}) = \dot{v}\,\underline{\hat{t}} + v\frac{d}{dt}\underline{\hat{t}} = \dot{v}\,\underline{\hat{t}} + v\underbrace{\frac{d\underline{\hat{t}}}{ds}}_{\frac{\hat{n}}{\hat{\varrho}}}\cdot\underbrace{\frac{ds}{dt}}_{v}$$

$$\boxed{\underline{a}(t) = \dot{v}\,\underline{\hat{t}} + \frac{v^2}{\hat{\varrho}}\underline{\hat{n}}}$$

liegt in der Schmiegungsebene

Tangential-Beschleunigung

Normal- (Zentripetal-) beschleunigung

Zusammenfassung (Frenet'sche Formeln)

$$\frac{d\underline{r}}{ds} = \underline{\hat{t}}$$

$$\frac{d}{ds}\underline{\hat{t}} = \kappa\,\underline{\hat{n}}$$

κ Krümmung $\kappa := \frac{d\underline{\hat{t}}}{ds}$

(i) $\frac{d}{ds}\underline{\hat{b}} = -\tau\,\underline{\hat{n}}$

τ = Torsion der Raumkurve

$\tau := \left|\frac{d\underline{\hat{b}}}{ds}\right|$

$$\underline{\hat{t}} \times \underline{\hat{n}} = \underline{\hat{b}}$$

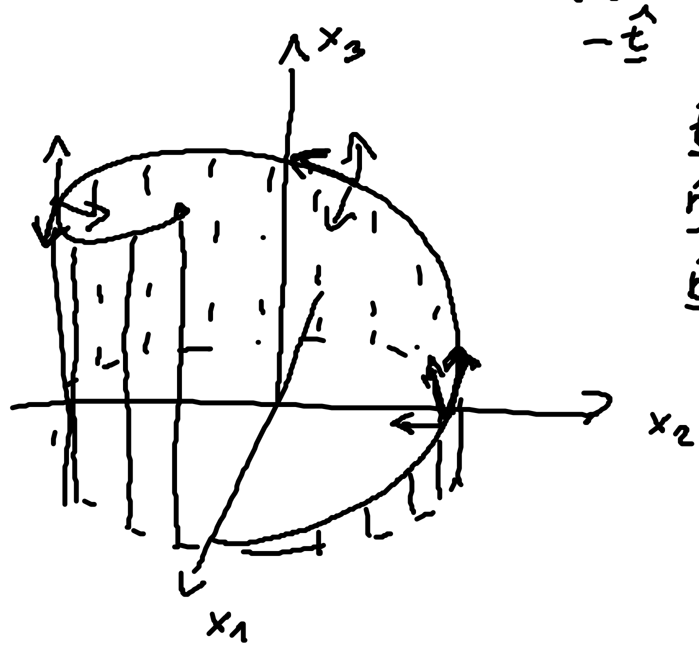
(ii) $\frac{d}{ds}\underline{\hat{n}} = -\kappa\,\underline{\hat{t}} + \tau\,\underline{\hat{b}}$

(„Beweis" (i) $\frac{d}{ds}\underline{\hat{b}} = \underbrace{\underbrace{\frac{d}{ds}\underline{\hat{t}}}_{\kappa\underline{\hat{n}}} \times \underline{\hat{n}}}_{0} + \underline{\hat{t}} \times \frac{d}{ds}\underline{\hat{n}} \quad \Rightarrow \quad \frac{d\underline{\hat{b}}}{ds} \perp \underline{\hat{t}} \qquad \frac{d\underline{\hat{b}}}{ds} \perp \underline{\hat{b}}$

$$\frac{d}{ds}\underline{\hat{b}} \parallel \underline{\hat{n}}$$

$$\frac{d}{ds}\underline{\hat{b}} = -\left|\frac{d\underline{b}}{ds}\right|\underline{\hat{n}}$$

(ii) $\frac{d}{ds}\underline{\hat{n}} = \frac{d\underline{\hat{b}}}{ds} \times \underline{\hat{t}} + \underline{\hat{b}} \times \frac{d\underline{\hat{t}}}{ds} = -\tau\,\underline{\hat{n}} \times \underline{\hat{t}} + \kappa\,\underbrace{\underline{\hat{b}} \times \underline{\hat{n}}}_{-\underline{\hat{t}}} = \tau\,\underline{\hat{b}} - \kappa\,\underline{\hat{t}}$)



4.2 Krummlinige Koordinaten

Motivation: Symmetrien ausnutzen, z.B. Kugelsymmetrie

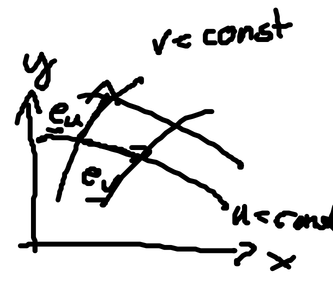
Sei $\underline{r} = \underline{r}(u, v, w)$ krummlinigen Koordinaten u, v, w

$$\Delta \underline{r} = \underline{r}(u+\Delta u, v+\Delta v, w+\Delta w) - \underline{r}(u, v, w) \qquad \text{(kleine Änderungen)}$$

$$= \frac{\partial \underline{r}}{\partial u}\Delta u + \frac{\partial \underline{r}}{\partial v}\Delta v + \frac{\partial \underline{r}}{\partial w}\Delta w \qquad \text{(Taylorentwicklung)}$$

Tangenten - Einheitsvektoren:

$$\underline{e}_u = \frac{\frac{\partial \underline{r}}{\partial u}}{\left|\frac{\partial \underline{r}}{\partial u}\right|} \qquad \underline{e}_v = \frac{\frac{\partial \underline{r}}{\partial v}}{\left|\frac{\partial \underline{r}}{\partial v}\right|} \qquad \underline{e}_w = \frac{\frac{\partial \underline{r}}{\partial w}}{\left|\frac{\partial \underline{r}}{\partial w}\right|}$$



$$\Rightarrow \Delta \underline{r} = \underline{e}_u \left|\frac{\partial \underline{r}}{\partial u}\right| \Delta u + \underline{e}_v \left|\frac{\partial \underline{r}}{\partial v}\right| \Delta v + \underline{e}_w \left|\frac{\partial \underline{r}}{\partial w}\right| \Delta w$$

Linienelement $\quad \Delta s := |\Delta \underline{r}|$

$$(\Delta s)^2 = \Delta \underline{r}\,\Delta \underline{r} = \underline{e}_u\,\underline{e}_u \left|\frac{\partial \underline{r}}{\partial u}\right|^2 (\Delta u)^2 + \underline{e}_u\,\underline{e}_v \left|\frac{\partial \underline{r}}{\partial u}\right| \left|\frac{\partial \underline{r}}{\partial v}\right| \Delta u \Delta v + \ldots$$

Zusammengefasst

$$(\Delta s)^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij}\, \Delta u_i\, \Delta u_j$$

$u_1 = u \quad u_2 = v \quad u_3\ w$

$$g_{ij} = \frac{\partial \underline{r}}{\partial u_i} \cdot \frac{\partial \underline{r}}{\partial u_j}$$ Metrik

(metrischer Tensor)

Speziell

<u>krummlinige - orthogonale Koordinaten</u>

$\underline{e}_{u_i} \cdot \underline{e}_{u_j} = \delta_{ij}$ $\Rightarrow$ $(\Delta s)^2 = g_{uu}(\Delta u)^2 + g_{vv}(\Delta v)^2 + g_{ww}(\Delta w)^2$

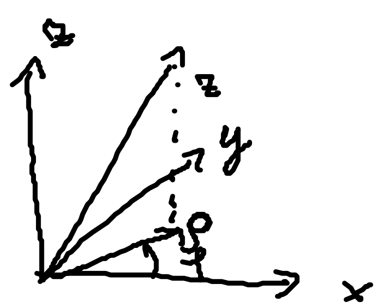
$=: \underset{\uparrow}{g_u^2}(\Delta u)^2 + g_v^2(\Delta v)^2 + g_w^2(\Delta w)^2$

metr Koeffizienten

$$g_u = \left|\frac{\partial \underline{r}}{\partial u}\right|$$

Beispiele:

a) Zylinderkoordinaten $\underline{r} = \underline{r}(\rho, \varphi, z)$



$x = \rho \cos\varphi$

$y = \rho \sin\varphi$

$z = z$

Umkehrung $\rho = \sqrt{x^2 + y^2}$

$\tan\varphi = y/x$

b) Kugelkoordinaten $\underline{r} = \underline{r}(r, \vartheta, \varphi)$

$x = r \sin\vartheta \cos\varphi$

$y = r \sin\vartheta \sin\varphi$

$z = r \cos\vartheta$