

Bem.: (i) Bestimmung von \underline{n} :

$$\text{Fläche } \alpha(x, y, z) = \text{const.} \Rightarrow \underline{n} = \frac{\nabla \alpha(x, y, z)}{|\nabla \alpha|}$$

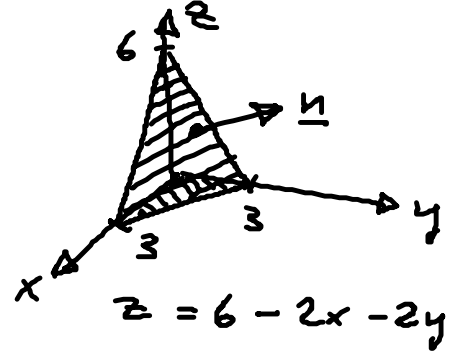
$$\text{z.B. } \alpha = x^2 + y^2 + z^2 = R^2 \Rightarrow \underline{n} = \frac{\underline{r}}{|\underline{r}|}$$

(ii) Verbleibendes Integral $\mathcal{I} = \iint dx dy f(x, y)$

$$\text{Sei } x_a \leq x \leq x_b$$

$$y = y(x)$$

$$\mathcal{I} = \int_{x_a}^{x_b} dx \int_{y_1(x)}^{y_2(x)} f(x, y) dy$$



Beispiel

Vektorfeld $\underline{A}(\underline{r}) = (4z, 1, 2x)$

Fläche $\alpha = 2x + 2y + z = 6 = \text{const.}$

$$\text{Normalenvektor } \underline{n} = \frac{\nabla \alpha}{|\nabla \alpha|} = \frac{1}{|\nabla \alpha|} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2^2 + 2^2 + 1}} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad \underline{n} \cdot \underline{e}_z = \frac{1}{3}$$

$$\Rightarrow \mathcal{I} = \int_F \underline{A} \cdot d\underline{f} = \iint \frac{\underline{A} \cdot \underline{n}}{|\underline{n} \cdot \underline{e}_z|} dx dy$$

$$= \iint (4(6 - 2x - 2y), 1, 2x) \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \cdot \frac{1}{3} \frac{dx dy}{\frac{1}{3}}$$

$$= \iint (50 - 14x - 16y) dx dy$$

Integrationsgrenzen: $z=0 \xrightarrow{(2x+2y+z=6)} x+y=3$

$$\mathcal{I} = \int_0^3 dx \int_0^{3-x} dy (50 - 14x - 16y)$$

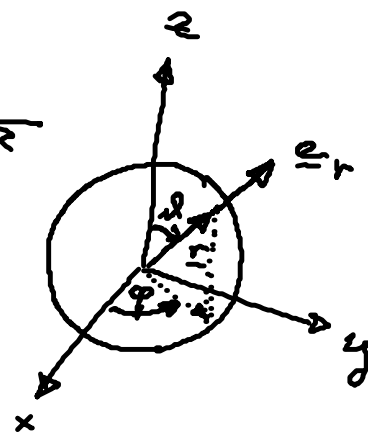
$$\begin{aligned}
 &= \int_0^3 dx \left[(50-14x)y - 8y^2 \right]_0^{3-x} \\
 &= \int_0^3 dx \left[(50-14x)(3-x) - 8(3-x)^2 \right] \\
 &= 90 \quad (?)
 \end{aligned}$$

Geschichten: symmetrieangepasste krummlinige Koordinaten

a) Kugelsymmetrie

$$\underline{r} = \begin{pmatrix} r \sin \varrho \cos \varphi \\ r \sin \varrho \sin \varphi \\ r \cos \varrho \end{pmatrix}$$

$$\begin{aligned}
 r &= \sqrt{x^2 + y^2 + z^2} \\
 \tan \varphi &= \frac{y}{x} \\
 \cos \varrho &= \frac{z}{r}
 \end{aligned}$$



Normalenvektor:

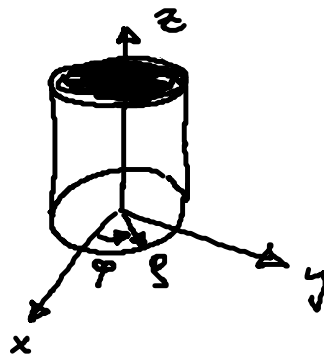
$$\underline{n} = \underline{e}_r = \frac{\underline{r}}{|\underline{r}|} = \begin{pmatrix} \sin \varrho \cos \varphi \\ \sin \varrho \sin \varphi \\ \cos \varrho \end{pmatrix}$$

radialer Einheitsvektor

b) Zylindersymmetrie

$$\underline{r} = \begin{pmatrix} \rho \cos \varphi \\ \rho \sin \varphi \\ z \end{pmatrix}$$

$$\begin{aligned}
 \rho &= \sqrt{x^2 + y^2} \\
 \tan \varphi &= \frac{y}{x}
 \end{aligned}$$



Transformation der Integrationsvariablen:

$$\iint \dots dx dy \rightarrow \iint \dots du dv$$

mit Funktionaldeterminante

z.B. $\left. \begin{matrix} u = \varrho \\ v = \varphi \end{matrix} \right\}$ bei Kugelsymmetrie

$$\frac{\partial(x,y)}{\partial(u,v)} := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0 \quad (\text{sonst Transformation nicht erlaubt})$$

$$\boxed{\iint f(x,y) dx dy = \iint f(x(u,v), y(u,v)) \frac{\partial(x,y)}{\partial(u,v)} du dv}$$

Beispiele: a) $(x, y) \rightarrow (r, \varphi)$ Kugeloberfläche

$$\frac{\partial(x, y)}{\partial(r, \varphi)} = R_0^2 \sin r \cos r, \quad \text{also } d\vec{f} = R_0^2 \sin r \cos r \, dr \, d\varphi$$

b) $(x, y) \rightarrow (\rho, \varphi)$ Zylinderoberfläche



$$\frac{\partial(x, y)}{\partial(\rho, \varphi)} = \rho, \quad \text{also } d\vec{f} = \rho \, d\rho \, d\varphi$$

Auswertung für Kugeloberfläche:

$$\mathcal{F} = \oint \underline{A} \cdot \underline{d\vec{f}} = \oint \frac{\underline{A} \cdot \underline{n}}{|\underline{n} \cdot \underline{e}_z|} \, dx \, dy = \int_0^{2\pi} d\varphi \int_0^\pi dr \sin r \cos r \, R_0^2 \frac{\underline{A} \cdot \underline{e}_r}{|\underline{n} \cdot \underline{e}_z|}$$

$$\underline{n} = \underline{e}_r = \begin{pmatrix} \sin r \cos \varphi \\ \sin r \sin \varphi \\ \cos r \end{pmatrix} \Rightarrow \underline{n} \cdot \underline{e}_z = \cos r$$

$$\mathcal{F} = \int_0^{2\pi} d\varphi \int_0^\pi dr \sin r \, R_0^2 (\underline{A} \cdot \underline{e}_r)$$

c) Auswertung von Flächenintegralen in Parameterdarstellung (s, t) :

$$\underline{r} = \underline{r}(s, t)$$

$$\mathcal{F} = \int \underline{A}(\underline{r}) \cdot \underline{d\vec{f}} = \underline{A}(\underline{r}(s, t)) \cdot \left(\frac{\partial \underline{r}}{\partial s} \times \frac{\partial \underline{r}}{\partial t} \right) \, ds \, dt$$

5.3.2 Volumenintegrale

$$I = \int dV \phi(\underline{r}) = \int d^3r \phi(\underline{r})$$

a) Kartes. Koord. $I = \int dx \int dy \int dz \phi(x, y, z)$

b) Krummlinige Koord. $I = \iiint \tilde{\phi}(u, v, w) \frac{\partial(x, y, z)}{\partial(u, v, w)} du dv dw$

Kugelkoord. $I = \iiint \phi(r, \vartheta, \varphi) r^2 \sin \vartheta d\vartheta d\varphi dr$

Zylinderkoord. $I = \iiint \phi(\rho, \varphi, z) \rho d\rho d\varphi dz$