5.3 Mehrdimensionale Integrale

Praktische Auswertung?

5.3.1 Flächenintegrale

z.B. $\psi = \int_F \underline{A}(\underline{r}) \cdot d\underline{f}$ $\qquad d\underline{f} = \underline{n}\, df$ Flächenelement

speziell: geschlossene Oberfläche $S = \partial V$

$$\psi = \oint_S \underline{A}(\underline{r}) \cdot d\underline{f}$$

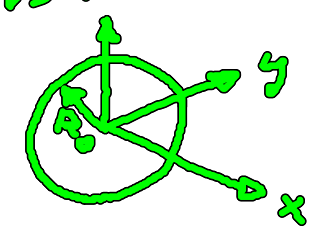
<u>Kartesische Koordinaten</u> $\underline{A} = \begin{pmatrix} A_x(x,y,z) \\ A_y(x,y,z) \\ A_z(x,y,z) \end{pmatrix}$

a) Parametrisierung der Fläche durch (x,y):

$$\underline{r} = (x, y, z(x,y))$$

z.B. Kugeloberfläche $x^2 + y^2 + z^2 = R_0^2$

$$\Rightarrow z = \pm\sqrt{R_0^2 - x^2 - y^2}$$



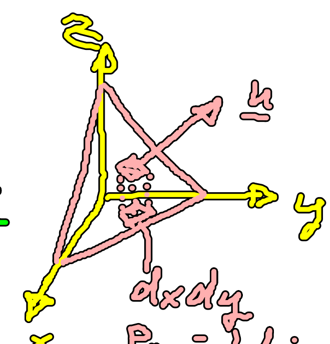
b) Bestimmung des vektoriellen Flächenelements

$$d\underline{f} = \underline{n}\, df$$

Es gilt: $|d\underline{f} \cdot \underline{e}_z| = |df\, \underline{n} \cdot \underline{e}_z| \overset{!}{=} dx\,dy$

$\underline{e}_z$: Einheitsvektor

$$\Leftrightarrow df = \frac{dx\,dy}{|\underline{n} \cdot \underline{e}_z|}$$



dxdy Projektion des Flächenstücks df auf (x,y)-Ebene

$$\Rightarrow \psi = \int \underline{A}(\underline{r}) \cdot d\underline{f} = \iint \frac{\underline{A}(x,y,z(x,y)) \cdot \underline{n}(x,y,z(x,y))}{|\underline{n} \cdot \underline{e}_z|}\, dx\,dy$$

Bem.: (i) Bestimmung von $\underline{n}$:

Fläche $\alpha(x,y,z) = \text{const.} \Rightarrow \underline{n} = \dfrac{\underline{\nabla}\alpha(x,y,z)}{|\underline{\nabla}\alpha|}$

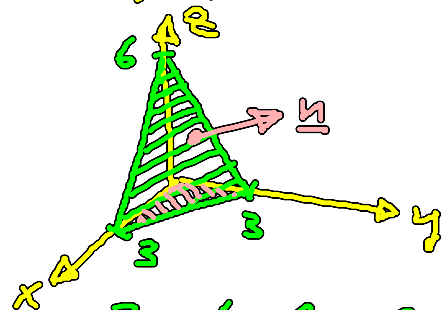
z.B. $\alpha = x^2+y^2+z^2 = R_0^2 \Rightarrow \underline{n} = \dfrac{\underline{r}}{|\underline{r}|}$

(ii) Verbleibendes Integral $\Psi = \iint dx\,dy\; f(x,y)$

Sei: $x_a \le x \le x_b$

$y = y(x)$

$$\Psi = \int_{x_a}^{x_b} dx \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy$$



$z = 6 - 2x - 2y$

Beispiel

Vektorfeld $\underline{A}(\underline{r}) = (4z, 1, 2x)$

Fläche $\alpha = 2x+2y+z = 6 = \text{const.}$

Normalenvektor $\underline{n} = \dfrac{\underline{\nabla}\alpha}{|\underline{\nabla}\alpha|} = \dfrac{1}{|\underline{\nabla}\alpha|}\begin{pmatrix}2\\2\\1\end{pmatrix}$

$$= \frac{1}{\sqrt{2^2+2^2+1}}\begin{pmatrix}2\\2\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2\\2\\1\end{pmatrix}, \quad \underline{n}\cdot\underline{e}_z = \frac{1}{3}$$

$$\Rightarrow \Psi = \int_F \underline{A}\cdot d\underline{f} = \iint \frac{\underline{A}\cdot\underline{n}\;dx\,dy}{|\underline{n}\cdot\underline{e}_z|}$$

$$= \iint (4(6-2x-2y), 1, 2x)\cdot\begin{pmatrix}2\\2\\1\end{pmatrix}\cdot\cancel{\frac{1}{3}}\;\frac{dx\,dy}{\cancel{\frac{1}{3}}}$$

$$= \iint (50 - 14x - 16y)\,dx\,dy$$

Integrationsgrenzen: $z = 0 \overset{(2x+2y+z=6)}{\Longrightarrow} x+y = 3$

$$\Psi = \int_0^3 dx \int_0^{3-x} dy\;(50 - 14x - 16y)$$

$$= \int_0^3 dx \left[(50-14x)y - 8y^2\right]_0^{3-x}$$

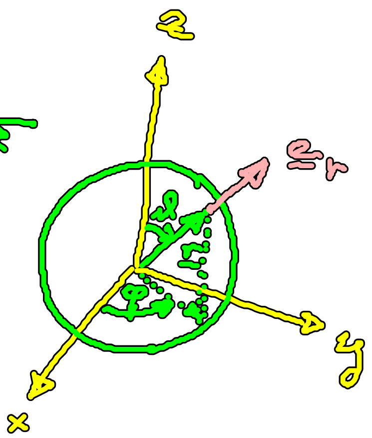
$$= \int_0^3 dx \left[(50-14x)(3-x) - 8(3-x)^2\right]$$

$$= 90 \quad (?)$$

Geschickter: symmetrieangepasste krummlinige Koordinaten

a) <u>Kugelsymmetrie</u>

$$\underset{\sim}{r} = \begin{pmatrix} r \sin\vartheta \cos\varphi \\ r \sin\vartheta \sin\varphi \\ r \cos\vartheta \end{pmatrix} \qquad r = \sqrt{x^2+y^2+z^2}, \quad \tan\varphi = \frac{y}{x}, \quad \cos\vartheta = \frac{z}{r}$$



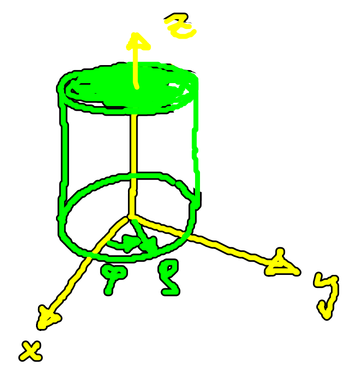
Normalenvektor:

$$\underset{\sim}{n} = \underset{\sim}{e}_r = \frac{\underset{\sim}{r}}{|\underset{\sim}{r}|} = \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}$$

radialer Einheitsvektor

b) <u>Zylindersymmetrie</u>

$$\underset{\sim}{r} = \begin{pmatrix} \rho \cos\varphi \\ \rho \sin\varphi \\ z \end{pmatrix} \qquad \rho = \sqrt{x^2+y^2}, \quad \tan\varphi = \frac{y}{x}$$



Transformation der Integrationsvariablen:

$$\iint \ldots dx\,dy \longrightarrow \iint \ldots du\,dv$$

mit Funktionaldeterminante

z.B. $\left.\begin{matrix} u = \vartheta \\ v = \varphi \end{matrix}\right\}$ bei Kugelsymmetrie

$$\frac{\partial(x,y)}{\partial(u,v)} := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0 \quad \text{(sonst Transformation nicht erlaubt)}$$

$$\boxed{\iint f(x,y)\,dx\,dy = \iint f(x(u,v), y(u,v))\,\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv}$$

Beispiele: a) $(x,y) \longrightarrow (\vartheta,\varphi)$ Kugeloberfläche

$$\frac{\partial(x,y)}{\partial(\vartheta,\varphi)} = R_0^2 \sin\vartheta \cos\vartheta \quad , \quad \text{also} \quad df = R_0^2 \sin\vartheta \cos\vartheta \, d\vartheta \, d\varphi$$

b) $(x,y) \rightarrow (\varrho,\varphi)$ Zylinderoberfläche

$$\frac{\partial(x,y)}{\partial(\varrho,\varphi)} = \varrho \quad , \quad \text{also} \quad df = \varrho \, d\varrho \, d\varphi$$

Auswertung für Kugeloberfläche:

$$\Phi = \oint \underline{A} \cdot d\underline{f} = \oint \frac{\underline{A}\cdot\underline{n}}{|\underline{n}\cdot\underline{e}_z|} dx\, dy = \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin\vartheta \cos\vartheta \, R_0^2 \frac{\underline{A}\cdot\underline{e}_r}{|\underline{n}\cdot\underline{e}_z|}$$

$$\underline{n} = \underline{e}_r = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} \Rightarrow \underline{n}\cdot\underline{e}_z = \cos\vartheta$$

$$\Phi = \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin\vartheta \, R_0^2 (\underline{A}\cdot\underline{e}_r)$$

c) Auswertung von Flächenintegralen in Parameterdarstellung (s,t):

$$\underline{r} = \underline{r}(s,t)$$

$$\Phi = \int \underline{A}(\underline{r})\cdot d\underline{f} = \underline{A}(\underline{r}(s,t))\cdot\left(\frac{\partial\underline{r}}{\partial s}\times\frac{\partial\underline{r}}{\partial t}\right) ds\, dt$$

5.3.2 Volumenintegrale

$$I = \int dV \, \phi(\underline{r}) = \int d^3r \, \phi(\underline{r})$$

a) Kartes. Koord. $I = \int dx \int dy \int dz\ \phi(x,y,z)$

b) Krummlinige Koord. $I = \iiint \tilde{\phi}(u,v,w) \frac{\partial(x,y,z)}{\partial(u,v,w)}\, du\, dv\, dw$

Kugelkoord. $I = \iiint \phi(r,\vartheta,\varphi)\, r^2 \sin\vartheta\, d\vartheta\, d\varphi\, dr$

Zylinderkoord. $I = \iiint \phi(\rho,\varphi,z)\, \rho\, d\rho\, d\varphi\, dz$