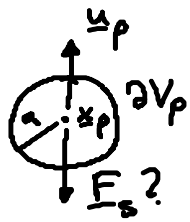
# 4.4. Stokes Reibung

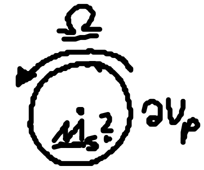
- 2 Standardsituationen:

(i) linear bewegte Kugel




Reibungskraft $\underline{F}_s$?

(ii) rotierende Kugel




Reibungsdrehmoment $\underline{M}_s$?

Berechne: $\underline{v}(\underline{x}),\ p(\underline{x}) \longrightarrow \underline{\underline{T}} \longrightarrow \underline{F}_s,\ \underline{M}_s$

## a) Translation

- Randbedingungen:

$$\left.\begin{aligned} &(i)\ \underline{v}(\underline{x}) \longrightarrow 0, && r = |\underline{x}| \longrightarrow \infty \\ &(ii)\ \underline{v}(\underline{x}) = \underline{u}_p, && \underline{x} \in \partial V_p \end{aligned}\right\} \quad (4.17)$$

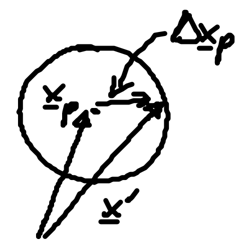
1. Weg: mit $\underline{\underline{O}}(\underline{x}-\underline{x}')$

Ansatz: Oberflächenkraftdichte von Kugel auf Flüssigkeit

$$\left. \varrho\,\underline{b}(\underline{x}') \right|_{\underline{x}' \in \partial V_p} = \frac{c}{4\pi a^2}\,\underline{u}_p \quad (4.18)$$

... konstant auf $\partial V_p$ !!

$$\longrightarrow \quad \underline{v}(\underline{x}) \overset{(4.6)}{=} \frac{c}{4\pi a^2} \oint_{\partial V_p} \underbrace{\underline{\underline{O}}(\underline{x}-\underline{x}')}_{\text{Lsg. der Stokes Gln. für } \underline{x}-\underline{x}' \neq 0} \underline{u}_p \, df' \qquad (4.19)$$




mit $\underline{\underline{O}}(\underline{x}-\underline{x}') = \underline{\underline{O}}(\underline{x} - (\underline{x}_p + \Delta\underline{x}_p))$

& Hilfsatz (4.5):

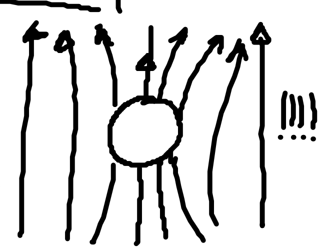
$$\left[\frac{1}{4\pi a^2} \oint_{\partial V_p} \underline{v}(\underline{x}) \, df = \left(1 + \frac{1}{6} a^2 \nabla_p^2\right) \underline{v}(\underline{x}_p)\right]$$

$$\longrightarrow \quad \underline{v}(\underline{x}) = c\left(1 + \frac{1}{6} a^2 \nabla_p^2\right) \underline{\underline{O}}(\underline{x}-\underline{x}_p) \underline{u}_p$$

R.B. (4.17) (ii) $\xrightarrow{\text{o.B.}}$ $c = 6\pi\eta a \qquad (4.20)$

$$\boxed{\begin{aligned} \underline{v}(\underline{x}) &= \underline{\underline{S}}(\underline{x}-\underline{x}_p) \, \underline{u}_p \\ \text{mit } \underline{\underline{S}}(\underline{x}) &= 6\pi\eta a \left(1 + \frac{1}{6} a^2 \nabla^2\right) \underline{\underline{O}}(\underline{x}) \\ &\overset{\text{o.B.}}{=} \frac{3}{4} \frac{a}{r}\left(\underline{\underline{1}} + \frac{\underline{x}\otimes\underline{x}}{r^2}\right) + \frac{1}{4}\left(\frac{a}{r}\right)^3 \left(\underline{\underline{1}} - 3\frac{\underline{x}\otimes\underline{x}}{r^2}\right) \end{aligned}} \qquad (4.21)$$

NB: für $r \gg a$: $\quad \underline{v}(\underline{x}) \approx \underline{\underline{O}}(\underline{x}-\underline{x}_p) \underbrace{6\pi\eta a \, \underline{u}_p}_{\underline{F} = -\underline{F}_s \; !}$ s.u.




... Stokeslet

2. Weg: Löse (4.1) direkt $\longrightarrow$ $\underline{v}(\underline{x}), p(\underline{x})$ [s. Übung]

• Stokessche Reibungskraft:

1. Weg: $\underline{F}_s \overset{\text{actio = reactio}}{=} - \int_{\partial V_p} \rho \, b(\underline{x}) \, df \overset{(4.18)}{=} - c \underline{u}_p \underbrace{\int \frac{df}{4\pi a^2}}_{1}$

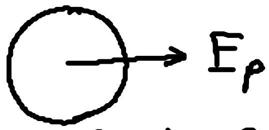
$$\longrightarrow \quad \boxed{\underline{F}_s = - 6\pi\eta a \underline{u}_p} \qquad (4.22)$$

... gültig für $t \gg \tau_H = \dfrac{a^2}{6\eta/\rho}$ !

2. Weg: $\underline{v}(\underline{x}), p(\underline{x}) \longrightarrow \underline{\underline{T}} = -p \, \underline{\underline{1}} + 2\eta \underline{\underline{A}}$

$$\longrightarrow \underline{F}_s = \int_{\partial V_p} \underline{\underline{T}}\, d\underline{f}$$

- **Brownsche Zeitskala** $\tau_B$:




$$\underline{u}_p(t=0) = \underline{0}$$

Bew.gl.: $m \dfrac{d\underline{u}_p}{dt} = \underline{F}_p + \underline{F}_s \longrightarrow m\dot{\underline{u}}_p + \gamma \underline{u}_p = \underline{F}_p$ (4.23) $\gamma = 6\pi\eta a$ ... Reibungskoeff.

$$\xrightarrow{\text{o.B.}} \boxed{\underline{u}_p = \frac{\underline{F}_p}{\gamma}\left(1 - e^{-t/\tau_m}\right) \quad \text{mit } \tau_m = \frac{m}{\gamma} = \frac{2}{9} a^2 \frac{\rho_p}{\eta}} \quad (4.24)$$

$m = \frac{4\pi}{3} a^3 \rho_p$

$\gamma = 6\pi\eta a$

vgl: $\tau_H = \frac{1}{6} a^2 \frac{\rho}{\eta}$

**Bsp:** $\rho_p = 1000 \frac{kg}{m^3}$, $a = 1\mu m$, $\eta = 10^{-3} \frac{kg}{ms} \longrightarrow \tau_m = \frac{2}{9} \cdot 10^{-6}\, s$

$\longrightarrow$ Impulsrelaxation vernachlässigbar auf Zeitskala

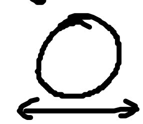
$$\boxed{\tau_B \gg \tau_H \geq \tau_m} \quad (4.25)$$

dann gilt: $$\boxed{\underline{u}_p(t) = \frac{1}{\gamma} \underline{F}_p(t)} \quad (4.26)$$

NB: (4.23) nicht konsistent, weil auf Zeiten $\tau_m \approx \tau_H$ keine stationäre Strömung existiert $\longrightarrow \zeta \underline{u}_p$ nicht gültig

$$\longrightarrow \text{besser: } \int_{-\infty}^{t} \underbrace{\zeta(t-t')}_{\text{Gedächtnis-fkt.}} \underline{u}_p(t')\,dt' \overset{\text{Kau-}}{\underset{\text{salität}}{=}} \int_{-\infty}^{\infty} \underbrace{\zeta(t-t')}_{=0,\ t'>t} \underline{u}_p(t')\,dt'$$

$$\xrightarrow[\text{satz}]{\text{Faltungs}} \underbrace{\zeta(\omega)}_{\text{frequenzabhängiger Reibungskoeff.}} \underline{u}_p(\omega) = \underline{F}_p(\omega)$$



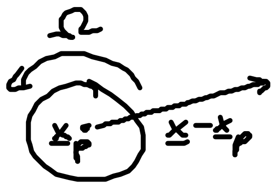
$$\underline{u}_p(t) = \underline{u}_p(\omega) e^{i\omega t}$$

$$\longrightarrow \underline{F}_p(\omega)$$

b) Rotation

- Randbed.: (i) $\underline{v}(\underline{x}) \longrightarrow 0, \quad r \to \infty$

  (ii) $\underline{v}(\underline{x}) = \underline{\Omega} \times (\underline{x} - \underline{x}_p), \quad \underline{x} \in \partial V_p$

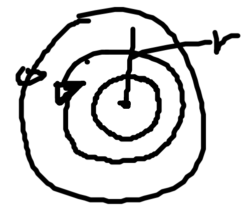



- 1. Weg: $\varsigma\, \underline{b}(\underline{x})\big|_{\partial V_p} = \frac{c}{4\pi a^2} \underline{\Omega} \times (\underline{x} - \underline{x}_p)$

  Integral: $\underline{v}(\underline{x}) = \frac{c}{4\pi a^2} \oint \underline{\underline{O}}(\underline{x}-\underline{x}')\, \varsigma\, [\underline{\Omega} \times (\underline{x}' - \underline{x}_p)]\, df'$

RB(ii) $\longrightarrow$ $c = 12\pi\eta a$

$$\longrightarrow \quad \boxed{\underline{v}(\underline{x}) = \left(\frac{a}{r}\right)^3 \underline{\Omega} \times (\underline{x} - \underline{x}_p) \sim \frac{1}{r^2} \quad , \; r = |\underline{x} - \underline{x}_p|} \quad (4.27)$$




- 2. Weg: s. Übungen

- Stokessches Drehmoment:

$$\underline{M}_s = -\int_{\partial V_p} [(\underline{x} - \underline{x}_p) \times \rho\, \underline{\sigma}(\underline{x})]\, df$$

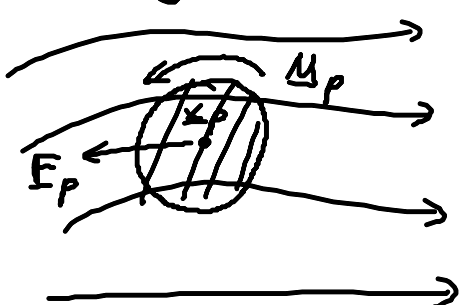
[Abstandsvektor × Kraft]

$$\xrightarrow{\text{z.B.}} \quad \boxed{\underline{M}_s = -8\pi\eta a^3 \underline{\Omega}} \quad (4.28)$$

## 4.5 Faxén-Theorem

- tiefere Einsicht, nur für kugelförmige Teilchen

- Problemstellung:




$\underline{v}_0(\underline{x})$ … Lsg. der Stokes-Gln. ohne Teilchen!

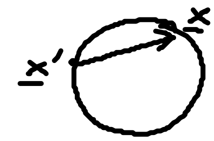
$\underline{u}_p, \underline{\Omega}_p$ ?

- Geschw. der Teilchenoberfläche: $\underline{x} \in \partial V_p$

$$\underbrace{\underline{u}_p + \underline{\Omega}_p \times (\underline{x} - \underline{x}_p)}_{(i)} = \underbrace{\underline{v}_0(\underline{x})}_{(iv)} + \int \underbrace{df'\, \underline{\underline{O}}(\underline{x} - \underline{x}')}_{(ii),\,(iii)} \underbrace{\rho\, \underline{\sigma}(\underline{x}')}_{\text{Oberflächenkraftdichte von Teilchen auf Flüssigkeit}} \quad (4.29)$$

Betrachte: $\frac{1}{4\pi a^2} \int_{\partial V_p} (4.29)\, df$

$$\longrightarrow \text{(i)} \int_{\partial V_p} (\underline{x} - \underline{x}_p)\, df = 0!$$

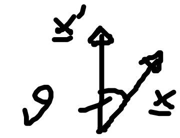



$$\text{(ii)} \frac{1}{4\pi a^2} \int_{\partial V_p} df\, \underline{\underline{O}}(\underline{x} - \underline{x}') \quad , \quad \underline{x}, \underline{x}' \in \partial V_p$$

$$\overset{!}{=} c\, \underline{\underline{1}} \quad (4.31)$$

$$\xrightarrow[(4.15)]{Sp\,(4.31)} \quad 3c = \frac{1}{4\pi a^2} \frac{4}{8\pi\eta} \int \frac{df}{|\underline{x} - \underline{x}'|} \overset{\text{Kugel}}{\underset{\text{Koord}}{=}} \frac{1}{2\pi\eta a}$$

$$\underline{\underline{O}}(\underline{x}) = \frac{1}{8\pi\eta r}\left(\underline{\underline{1}} + \frac{\underline{x} \circ \underline{x}}{r^2}\right)$$




$$\longrightarrow \frac{1}{4\pi a^2} \int_{\partial V_p} df\, \underline{\underline{O}}(\underline{x} - \underline{x}') = c\, \underline{\underline{1}} = \frac{1}{6\pi\eta a} \quad (4.32)$$

$$\text{(iii)} \frac{1}{6\pi\eta a} \int_{\partial V_p} df' \underline{\sigma}^{b}(\underline{x}') = \frac{1}{6\pi\eta a} \underline{F}_p \quad (4.33)$$

$$\text{(iv)} \frac{1}{4\pi a^2} \int_{\partial V_p} \underline{v}_0(\underline{x})\, df \underset{(4.5)}{=} \left(1 + \frac{1}{6} a^2 \nabla_p^2\right) \underline{v}_0(\underline{x}_p) \quad (4.34)$$

$$\xrightarrow[(4.29)]{\text{(i)}-\text{(iv) in}} \boxed{\underline{u}_p = \frac{1}{6\pi\eta a} \underline{F}_p + \left(1 + \frac{1}{6} a^2 \nabla_p^2\right) \underline{v}_0(\underline{x}_p)} \quad (4.35)$$

... Faxén-Theorem für Translation

(i) $\underline{v}_0(\underline{x}) = 0 \longrightarrow$ Stokes (4.22)

$$\underline{F}_p = -\underline{F}_S$$

(ii) kräftefreies Teilchen:

$$\underline{F}_p = 0 \longrightarrow \underline{u}_p = \underline{v}_0(\underline{x}_p) + \frac{1}{6} a^2 \nabla_p^2 \underline{v}_0(\underline{x}_p) \;!$$

(iii) wichtig für Störungstheorie für HW [s. Kap. 6]

- Faxén-Theorem für Rotation: $\left[ \int (\underline{x} \times \underline{x}_p) \times (4.29)\, df \longrightarrow \ldots \right]$

$$\boxed{\underline{\Omega}_p = \frac{1}{8\pi\eta a^3}\underline{M}_p + \underbrace{\frac{1}{2}\nabla_p \times \underline{u}_0(\underline{x}_p)}_{\text{Vortex in Strömung!}}} \quad (4.36)$$

[vgl. Gl. (3.9)]