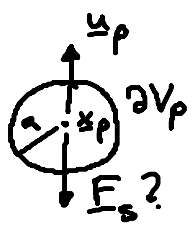
4.4. Stokes Reibung

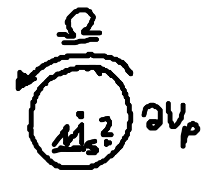
- 2 Standardsituationen:

(i) linear bewegte Kugel



Reibungskraft $\underline{F}_s$?

(ii) rotierende Kugel



Reibungsdrehmoment $\underline{M}_s$?

Berechne: $\underline{v}(\underline{x}),\ p(\underline{x}) \longrightarrow \underline{\underline{T}} \longrightarrow \underline{F}_s,\ \underline{M}_s$

a) Translation

- Randbedingungen:

$$\left.\begin{array}{lll} \text{(i)} & \underline{v}(\underline{x}) \longrightarrow 0, & r = |\underline{x}| \longrightarrow \infty \\ \text{(ii)} & \underline{v}(\underline{x}) = \underline{u}_p, & \underline{x} \in \partial V_p \end{array}\right\} \quad (4.17)$$

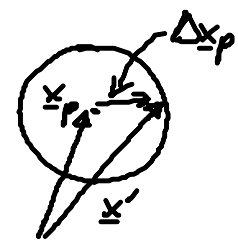
1. Weg: mit $\underline{\underline{O}}(\underline{x}-\underline{x}')$

Ansatz: Oberflächenkraftdichte von Kugel auf Flüssigkeit

$$\varsigma \underline{b}(\underline{x}')\Big|_{\underline{x}' \in \partial V_p} = \frac{c}{4\pi a^2}\, \underline{u}_p \quad (4.18)$$

... konstant auf ∂V_p !!

$$\longrightarrow \quad \underline{v}(\underline{x}) \overset{(4.6)}{=} \frac{c}{4\pi a^2} \oint_{\partial V_p} \underbrace{\underline{\underline{O}}(\underline{x}-\underline{x}')}_{\substack{\text{Lsg. der Stokes Gln.} \\ \text{für } \underline{x}-\underline{x}' \neq 0}} \underline{u}_p \, df' \qquad (4.19)$$



$$\text{mit } \underline{\underline{O}}(\underline{x}-\underline{x}') = \underline{\underline{O}}(\underline{x}-(\underline{x}_p+\Delta\underline{x}_p))$$

& Hilfssatz (4.5):

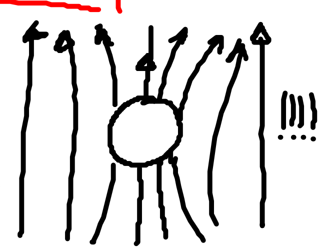
$$\left[\frac{1}{4\pi a^2} \oint_{\partial V_p} \underline{v}(\underline{x})\, df = \left(1 + \frac{1}{6} a^2 \nabla_p^2\right) \underline{v}(\underline{x}_p)\right]$$

$$\longrightarrow \quad \underline{v}(\underline{x}) = c\left(1 + \frac{1}{6} a^2 \nabla_p^2\right) \underline{\underline{O}}(\underline{x}-\underline{x}_p)\, \underline{u}_p$$

R.B. (4.17) (ii) $\xrightarrow{\text{o.B.}}$ $c = 6\pi\eta a$ (4.20)

$$\boxed{\begin{aligned} &\underline{v}(\underline{x}) = \underline{\underline{S}}(\underline{x}-\underline{x}_p)\, \underline{u}_p \\ &\text{mit } \underline{\underline{S}}(\underline{x}) = 6\pi\eta a \left(1 + \frac{1}{6} a^2 \nabla^2\right) \underline{\underline{O}}(\underline{x}) \\ &\qquad \overset{\text{o.B.}}{=} \frac{3}{4} \frac{a}{r} \left(\underline{\underline{1}} + \frac{\underline{x}\otimes\underline{x}}{r^2}\right) + \frac{1}{4}\left(\frac{a}{r}\right)^3 \left(\underline{\underline{1}} - 3\, \frac{\underline{x}\otimes\underline{x}}{r^2}\right) \end{aligned}} \qquad (4.21)$$

NB: für $r \gg a$: $\underline{v}(\underline{x}) \approx \underline{\underline{O}}(\underline{x}-\underline{x}_p) \underbrace{6\pi\eta a\, \underline{u}_p}_{\underline{F} = -\underline{F}_s}$! s.u.



... Stokeslet

2. Weg: Löse (4.1) direkt $\longrightarrow$ $\underline{v}(\underline{x}), p(\underline{x})$ [s. Übung]

- <u>Stokessche Reibungskraft:</u>

1. Weg: $\underline{F}_s \overset{\substack{\text{actio=} \\ \text{reactio}}}{=} - \int_{\partial V_p} \rho\, b(\underline{x})\, df \overset{(4.18)}{=} - c\, \underline{u}_p \underbrace{\int \frac{df}{4\pi a^2}}_{1}$

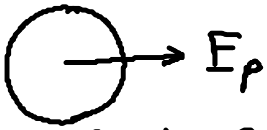
$$\longrightarrow \quad \boxed{\underline{F}_s = -6\pi\eta a\, \underline{u}_p} \qquad (4.22)$$

... gültig für $t \gg \tau_H = \frac{a^2}{6\eta/\rho}$!

2. Weg: $\underline{v}(\underline{x}), p(\underline{x}) \longrightarrow \underline{\underline{T}} = -p\, \underline{\underline{1}} + 2\eta \underline{\underline{A}}$

$$\longrightarrow \underline{F}_s = \int_{\partial V_p} \underline{\underline{T}}\, d\underline{f}$$

- <u>Brownsche Zeitskala</u> τ_B:



$\underline{u}_p(t=0) = \underline{0}$

Bew.gl.: $m\frac{d\underline{u}_p}{dt} = \underline{F}_p + \underline{F}_s \longrightarrow m\dot{\underline{u}}_p + \gamma\underline{u}_p = \underline{F}_p$ (4.23)

$\gamma = 6\pi\eta a$ … Reibungskoeff.

$$\xrightarrow{\text{o.B.}} \boxed{\begin{aligned} \underline{u}_p &= \frac{\underline{F}_p}{\gamma}\left(1 - e^{-t/\tau_m}\right) \\ \text{mit } \tau_m &= \frac{m}{\gamma} = \frac{2}{9} a^2 \frac{\rho_p}{\eta} \end{aligned}} \quad (4.24)$$

$m = \frac{4\pi}{3} a^3 \rho_p$

$\gamma = 6\pi\eta a$

vgl: $\tau_H = \frac{1}{6} a^2 \frac{\rho}{\eta}$

<u>Bsp:</u> $\rho_p = 1000 \frac{kg}{m^3}$, $a = 1\mu m$, $\eta = 10^{-3} \frac{kg}{ms} \longrightarrow \tau_m = \frac{2}{9} \cdot 10^{-6} s$

$\longrightarrow$ Impulsrelaxation vernachlässigbar auf Zeitskala

$$\boxed{\tau_B \gg \tau_H \geq \tau_m} \quad (4.25)$$

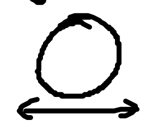
dann gilt: $\boxed{\underline{u}_p(t) = \frac{1}{\gamma} \underline{F}_p(t)}$ (4.26)

NB: (4.23) nicht konsistent, weil auf Zeiten $\tau_m \approx \tau_H$

keine stationäre Strömung existiert $\longrightarrow$ $\zeta \underline{u}_p$ nicht gültig

$$\longrightarrow \text{besser: } \int_{-\infty}^{t} \underbrace{\zeta(t-t')}_{\text{Gedächtnis-fkt.}} \underline{u}_p(t')\,dt' \overset{\text{Kausalität}}{=} \int_{-\infty}^{\infty} \underbrace{\zeta(t-t')}_{=0,\ t'>t} \underline{u}_p(t')\,dt'$$

$$\xrightarrow[\text{satz}]{\text{Faltungs}} \underbrace{\zeta(\omega)}_{\text{frequenzabhängiger Reibungskoeff.}} \underline{u}_p(\omega) = \underline{F}_p(\omega)$$



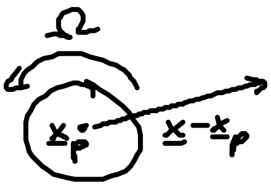
$$\underline{u}_p(t) = \underline{u}_p(\omega)\, e^{i\omega t}$$

$$\longrightarrow \underline{F}_p(\omega)$$

b) Rotation

- Randbed.: (i) $\underline{v}(\underline{x}) \longrightarrow 0, \quad r \to \infty$

  (ii) $\underline{v}(\underline{x}) = \underline{\Omega} \times (\underline{x} - \underline{x}_p), \quad \underline{x} \in \partial V_p$

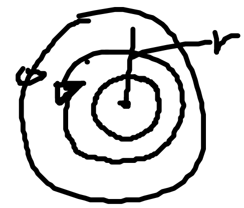


- 1. Weg: $S\ \underline{b}(\underline{x})\big|_{\partial V_p} = \frac{\underline{c}}{4\pi a^2}\, \underline{\Omega} \times (\underline{x} - \underline{x}_p)$

  ⋮ Integral: $\underline{v}(\underline{x}) = \frac{\underline{c}}{4\pi a^2} \oint \underline{\underline{O}}(\underline{x}-\underline{x}') \cdot [\underline{\Omega} \times (\underline{x}' - \underline{x}_p)]\, df'$

$$RB(ii) \longrightarrow c = 12\pi\eta a$$

$$\longrightarrow \quad \boxed{\underline{v}(\underline{x}) = \left(\frac{a}{r}\right)^3 \underline{\Omega} \times (\underline{x} - \underline{x}_p) \sim \frac{1}{r^2} \,, \quad r = |\underline{x} - \underline{x}_p|} \qquad (4.27)$$



- 2. Weg: s. Übung

- Stokessches Drehmoment:

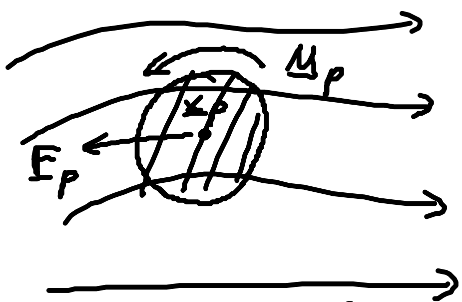
$$\underline{M}_s = - \int_{\partial V_p} [(\underline{x} - \underline{x}_p) \times \underline{\underline{\sigma}}(\underline{x})] \, df$$

[Abstandsvektor × Kraft]

$$\xrightarrow{z.B.} \quad \boxed{\underline{M}_s = -8\pi\eta a^3 \underline{\Omega}} \qquad (4.28)$$

4.5 Faxén-Theorem

- tiefere Einsicht, nur für kugelförmige Teilchen
- Problemstellung:



$\underline{v}_0(\underline{x})$ ... Lsg. der Stokes-Gln. ohne Teilchen!

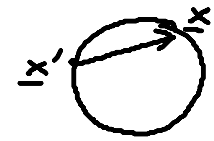
$\underline{u}_p, \underline{\Omega}_p$?

- Geschw. der Teilchenoberfläche: $\underline{x} \in \partial V_p$

$$\underbrace{\underline{u}_p + \underline{\Omega}_p \times (\underline{x} - \underline{x}_p)}_{(i)} = \underbrace{\underline{v}_0(\underline{x})}_{(iv)} + \int \underbrace{df' \, \underline{\underline{O}}(\underline{x} - \underline{x}')}_{(ii),(iii)} \underbrace{\underline{\underline{\sigma}}(\underline{x}')}_{\text{Oberflächenkraftdichte von Teilchen auf Flüssigkeit}} \qquad (4.29)$$

Betrachte: $\displaystyle \frac{1}{4\pi a^2} \int_{\partial V_p} (4.29) \, df$

$\longrightarrow$ (i) $\int_{\partial V_p} (\underline{x}-\underline{x}_p)\, df = 0!$

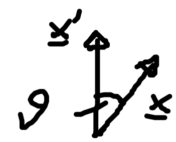


(ii) $\frac{1}{4\pi a^2}\int_{\partial V_p} df\, \underline{\underline{O}}(\underline{x}-\underline{x}')$, $\underline{x},\underline{x}' \in \partial V_p$

$\stackrel{!}{=} c\,\underline{\underline{1}}$ (4.31)

$\xrightarrow[(4.15)]{Sp\,(4.31)}$ $3c = \frac{1}{4\pi a^2}\,\frac{4}{8\pi\eta}\int \frac{df}{|\underline{x}-\underline{x}'|} \underset{\text{Koord}}{\overset{\text{Kugel}}{=}} \frac{1}{2\pi\eta a}$

$\underline{\underline{O}}(\underline{x}) = \frac{1}{8\pi\eta r}\left(\underline{\underline{1}}+\frac{\underline{x}\circ\underline{x}}{r^2}\right)$



$$\longrightarrow \frac{1}{4\pi a^2}\int_{\partial V_p} df\, \underline{\underline{O}}(\underline{x}-\underline{x}') = c\,\underline{\underline{1}} = \frac{1}{6\pi\eta a} \quad (4.32)$$

(iii) $\frac{1}{6\pi\eta a}\int_{\partial V_p} df'\, \underline{g}^{b}(\underline{x}') = \frac{1}{6\pi\eta a}\,\underline{F}_p$ (4.33)

(iv) $\frac{1}{4\pi a^2}\int_{\partial V_p} \underline{v}_0(\underline{x})\, df \underset{(4.5)}{=} \left(\underline{\underline{1}} + \frac{1}{6}a^2\nabla_p^2\right)\underline{v}_0(\underline{x}_p)$ (4.34)

$\xrightarrow[(4.29)]{(i)-(iv)\ \text{in}}$ $\boxed{\underline{u}_p = \frac{1}{6\pi\eta a}\,\underline{F}_p + \left(\underline{\underline{1}}+\frac{1}{6}a^2\nabla_p^2\right)\underline{v}_0(\underline{x}_p)}$ (4.35)

... Faxén-Theorem für Translation

(i) $\underline{v}_0(\underline{x}) = 0 \longrightarrow$ Stokes (4.22)
$\underline{F}_p = -\underline{F}_S$

(ii) kräftefreies Teilchen:
$\underline{F}_p = 0 \longrightarrow \underline{u}_p = \underline{v}_0(\underline{x}_p) + \frac{1}{6}a^2\nabla_p^2\,\underline{v}_0(\underline{x}_p)$!

(iii) wichtig für Störungstheorie für HW [s. Kap. 6]

- Faxén-Theorem für Rotation: $\left[\int (\underline{x}\times\underline{x}_p)\times(4.29)\, df \longrightarrow \dots\right]$

$$\boxed{\underline{\Omega}_p = \frac{1}{8\pi\eta a^3}\underline{M}_p + \underbrace{\frac{1}{2}\nabla_p \times \underline{u}_0(x_p)}_{\text{Vortex in Strömung!}}} \qquad (4.36)$$

[vgl. Gl. (3.9)]