4. Elektromagnetische Wellen

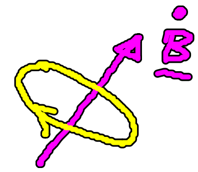
Im stat. Fall sind $\underline{E}$ und $\underline{B}$ entkoppelt

Im dynam. Fall sind $\underline{E}$ und $\underline{B}$ über

Verschieb.strom $\frac{1}{\mu_0}\underline{\nabla}\times\underline{B} - \underline{j} = \epsilon_0 \dot{\underline{E}}$

und das Induktionsgesetz

$$\underline{\nabla}\times\underline{E} = -\dot{\underline{B}}$$



gekoppelt

$\Rightarrow$ elektromagn. Wellenausbreitung

4.1 Freie Wellenausbreitung im Vakuum

ohne Quellen: $\rho = 0$, $\underline{j} = 0$

$$\Box\phi = -\frac{1}{\epsilon_0}\rho \qquad \Box\underline{A} = -\mu_0\underline{j} \quad \Rightarrow \quad \boxed{\begin{array}{l}\Box\phi = 0 \\ \Box\underline{A} = 0\end{array}}$$

homogene Wellengl.

Lorentz-Eichung

$\left(\underline{\nabla}\cdot\underline{A} + \epsilon_0\mu_0\frac{\partial}{\partial t}\phi = 0\right)$

$$\Box := \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

Wegen $\underline{E} = -\dot{\underline{A}} - \underline{\nabla}\phi$, $\underline{B} = \underline{\nabla}\times\underline{A}$

gilt auch

$$\boxed{\begin{aligned} \Box \underline{E} &= 0 \\ \Box \underline{B} &= 0 \end{aligned}}$$

$$\Big(\underline{\nabla} \times \underline{B} = \varepsilon_0 \mu_0 \dot{\underline{E}} \,, \quad \underline{\nabla} \times \underline{E} = - \dot{\underline{B}} \;:$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = \underline{\nabla} \underbrace{(\underline{\nabla} \cdot \underline{E})}_{0} - \Delta \underline{E} = - \underline{\nabla} \times \dot{\underline{B}} = - \varepsilon_0 \mu_0 \ddot{\underline{E}}$$

$$\Big(\Delta - \underbrace{\varepsilon_0 \mu_0}_{1/c^2} \frac{\partial^2}{\partial t^2} \Big) \underline{E} = 0 \quad \Big)$$

Allg. Lösung von $\Box u(\underline{r}, t) = 0$

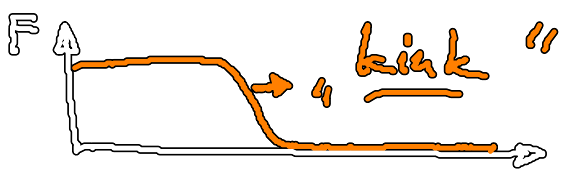
$$u(\underline{r}, t) = F(\underline{k} \cdot \underline{r} - \omega t)$$

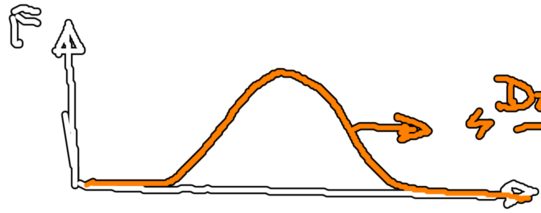
mit bel. 2× diff'barer Fkt. $F(\varphi)$

und $\boxed{\omega = c|\underline{k}|}$ (d'Alembert'sche Lösung)

Beweis $\quad \Box F = \left(\underline{k}^2 - \frac{\omega^2}{c^2} \right) F''(\varphi) = 0 \quad \Box$

NB: F muss nicht periodisch sein, z. B. solitäre Wellen:

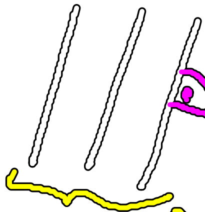




$\underline{k}$ Wellenvektor $\quad |\underline{k}| = \frac{2\pi}{\lambda}$

ω Frequenz

φ Phase



$\rightarrow \underline{k}$ (Ausbreit. richtung)

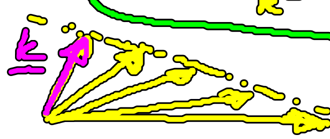
$\nabla\varphi(\underline{r},t) = \underline{k}$

ebene Welle

Flächen konst. Phase:

$\underline{k}\cdot\underline{r} - \omega t = \varphi(\underline{r},t) = \text{const.}$: Ebene

$$\underline{k}\cdot\underbrace{\left[\underline{r} - \frac{\underline{k}}{k^2}(\omega t + \varphi)\right]}_{\underline{r}(t) = \frac{1}{k^2}\underline{k}(\omega t+\varphi)} = 0$$



$\Rightarrow$ Phasengeschwind.: $\underline{v}_{ph} = \left.\frac{d\underline{r}}{dt}\right|_{\varphi=\text{const}} = \frac{\underline{k}}{k^2}\omega = c\underbrace{\frac{\underline{k}}{k}}_{=:\,\underline{n}}$

$=:\underline{n}$ Normalenvektor

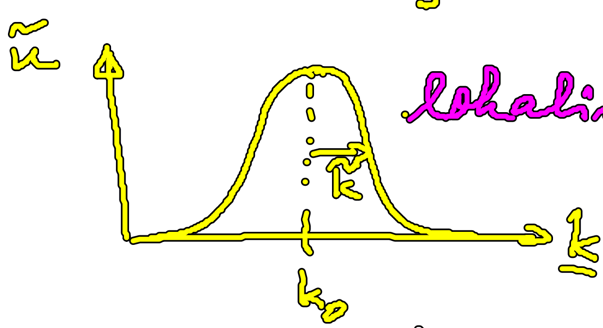
$$\boxed{v_{ph} = \frac{\omega}{k}}$$

Spezielle Lösung: harmon. ebene Welle

$$u(\underline{r},t) = \underbrace{\tilde{u}(\underline{k})}_{\text{komplexe Amplitude}}\, e^{i(\underline{k}\cdot\underline{r} - \omega t)}$$

Lineare Superposition (für allg. Dispersionsrel $\omega(\underline{k})$):

$$u(\underline{r},t) = \int d^3k\, \tilde{u}(\underline{k})\, e^{i(\underline{k}\cdot\underline{r} - \omega(\underline{k})t)}$$



lokalisiert in $\underline{k}$

$\Rightarrow$ Wellenpaket (im Ortsraum lokalisiert)

Denn: Taylor-Entwicklung der Phase um $\underline{k}_0$:

$$\omega(\underline{k}) \approx \underbrace{\omega(\underline{k}_0)}_{\omega_0} + \underbrace{(\underline{k}-\underline{k}_0)}_{\tilde{\underline{k}}}\, \underbrace{\nabla_{\underline{k}}\,\omega(\underline{k})\Big|_{\underline{k}=\underline{k}_0}}_{\underline{v}_g} + \dots$$

$$= \omega_0 + \underbrace{(\underline{k}-\underline{k}_0)}_{\tilde{\underline{k}}}\,\underline{v}_g$$

ergibt

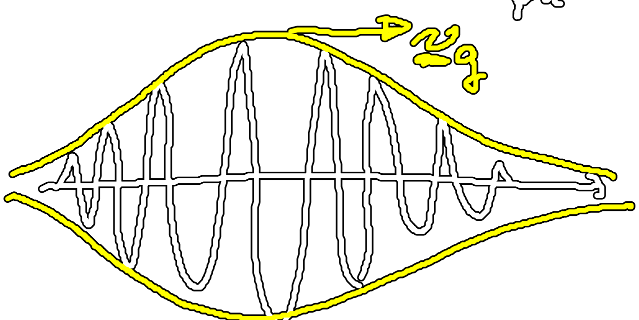
$$u(\underline{r},t) = \underbrace{e^{i(\underline{k}_0\underline{r}-\omega_0 t)}}_{\substack{\text{Trägerwelle}\\ \text{mit Phasen-}\\ \text{geschw.}\\ v_{ph}=\frac{\omega_0}{k_0}}}\ \underbrace{\int d^3\tilde{\underline{k}}\ \tilde{u}(\underline{k}_0+\tilde{\underline{k}})\, e^{i\tilde{\underline{k}}(\underline{r}-\underline{v}_g t)}}_{\substack{\text{Einhüllende}\\ \text{Max bewegt sich}\\ \text{Gruppengeschw.}}}$$

Trägerwelle mit Phasengeschw. $v_{ph} = \frac{\omega_0}{k_0}$

Einhüllende Max bewegt sich

Gruppengeschw.

$$\boxed{\underline{v}_g = \nabla_{\underline{k}}\,\omega(\underline{k})}$$



Dispersionsrelation $\omega(\underline{k})$

el. magn. Wellen im Vakuum $\omega(\underline{k}) = c|\underline{k}|$

$$\Rightarrow \underline{v}_g = c\,\frac{\underline{k}}{k} \overset{!}{=} \underline{v}_{ph} = \frac{1}{\sqrt{\varepsilon_0\mu_0}}\,\underline{n}$$

keine Dispersion (d.h. kein Zerfließen)!

(im Gegensatz zu el. magn. Wellen in dispersiven Medien oder qu. Materiewellen im Vakuum)

Polarisation

Betrachte $\underline{E}(\underline{r},t) = \underline{E}_0\, e^{i(\underline{k}\cdot\underline{r}-\omega t)}$

$\underline{B}(\underline{r},t) = \underline{B}_0\, e^{i(\underline{k}\cdot\underline{r}-\omega t)}$

Allg. gilt

$\underline{E}$ heißt <u>transversal</u>, falls $\underline{\nabla}\cdot\underline{E} = 0$ (quellenfrei)

$\Rightarrow i\underline{k}\cdot\underline{E} = 0 \Rightarrow \boxed{\underline{E} \perp \underline{k}}$

$\underline{E}$ heißt <u>longitudinal</u>, falls $\underline{\nabla}\times\underline{E} = 0$ (wirbelfrei)

$\Rightarrow i\underline{k}\times\underline{E} = 0 \quad \boxed{\underline{E} \parallel \underline{k}}$

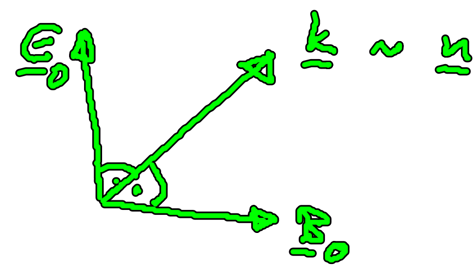
Für $\rho = 0$ ist wegen $\underline{\nabla}\cdot\underline{E} = 0$: $\underline{E}(\underline{r},t)$ transversal

stets wegen $\underline{\nabla}\cdot\underline{B} = 0$: $\underline{B}(\underline{r},t)$ transversal

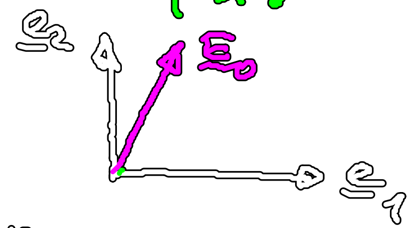
Weiter folgt aus $\underline{\nabla}\times\underline{E} + \dot{\underline{B}} = 0$:

$$\big(i\underline{k}\times\underline{E}_0 - i\underbrace{\omega}_{c|\underline{k}|}\underline{B}_0\big)\, e^{i(\underline{k}\cdot\underline{r}-\omega t)} = 0$$

$$\Rightarrow \boxed{\underline{B}_0 = \frac{1}{c}\,\underline{n}\times\underline{E}_0} \qquad \underline{n} := \frac{\underline{k}}{|\underline{k}|}$$



Richtung von $\mathrm{Re}\{\underline{E}_0, \underline{B}_0\}$ legt **Polarisation** fest:



$\underline{k} \parallel \underline{e}_3$-Achse

$$\underline{E}_0 = E_{01}\,\underline{e}_1 + E_{02}\,\underline{e}_2$$

$$\text{mit} \quad E_{0i} = a_i e^{i\delta_i} \in \mathbb{C}$$

Phys. Feld:
$$E_1(\underline{x},t) = \mathrm{Re}\left\{a_1 e^{i(\delta_1 + \underbrace{\underline{k}\,\underline{x} - \omega t}_{\varphi})}\right\} = a_1 \cos(\varphi + \delta_1)$$

$$E_2(\underline{x},t) = \mathrm{Re}\left\{a_2 e^{i(\delta_2 + \varphi)}\right\} = a_2 \cos(\varphi + \delta_2)$$

$$\frac{E_1}{a_1} = \cos\varphi \cos\delta_1 - \underline{\sin\varphi \sin}\,\delta_1$$

$$\frac{E_2}{a_2} = \cos\varphi \cos\delta_2 - \underline{\sin\varphi \sin}\,\delta_2$$

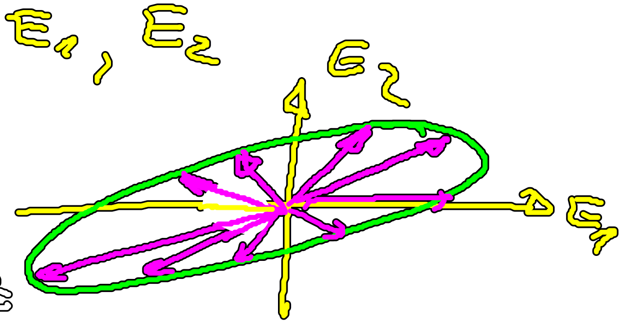
$$\frac{E_1}{a_1}\sin\delta_2 - \frac{E_2}{a_2}\sin\delta_1 = \cos\varphi \; \sin\underbrace{(\delta_2 - \delta_1)}_{\delta} \qquad (1)$$

$$\frac{E_1}{a_1}\cos\delta_2 - \frac{E_2}{a_2}\cos\delta_1 = \sin\varphi \; \sin\underbrace{(\delta_2 - \delta_1)}_{\delta} \qquad (2)$$

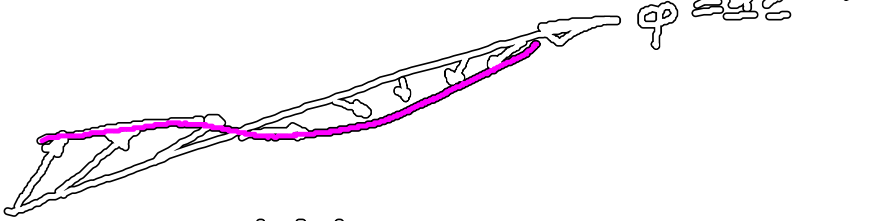
$(1)^2 + (2)^2$

$$\Longrightarrow \boxed{\left(\frac{E_1}{a_1}\right)^2 + \left(\frac{E_2}{a_2}\right)^2 - 2\frac{E_1}{a_1}\frac{E_2}{a_2}\cos\delta = \sin^2\delta}$$

Ellipsengl. für E_1, E_2



elliptische Polarisation



Spezialfälle

(a) linear polarisierte Welle : $\delta_1 = \delta_2 + n\pi$

$\sin\delta = 0, \quad \cos\delta = \pm 1$

$$\frac{E_1}{a_1} \pm \frac{E_2}{a_2} = 0$$

Gerade $\quad \underline{E}(\underline{r},t) = \underset{\text{reell}}{\underline{E}_0} \cos\varphi(\underline{r},t)$

(b) zirkular polaris. Welle $: a_1 = a_2 \equiv a$

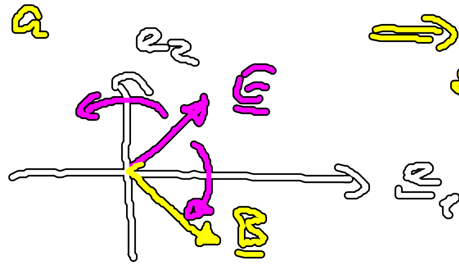
$\delta_1 = \delta_2 + (2n+1)\frac{\pi}{2}$

$\Longrightarrow \cos\delta = 0$

$\sin\delta = \pm 1$

$$E_1^2 + E_2^2 = a^2$$

Kreis



links- / rechts-zirkular polarisiert

Energiedichte
$$w = \frac{\varepsilon_0}{2}\underline{E}^2 + \frac{1}{2\mu_0}\underline{B}^2$$
$$= \frac{\varepsilon_0}{2}\underline{E}^2 + \frac{1}{2\mu_0}\frac{1}{c^2}\underline{E}^2$$
$$= 2\cdot\frac{\varepsilon_0}{2}\underline{E}^2$$

Energiestromdichte
$$\underline{S} = \frac{1}{\mu_0}\underline{E}\times\underline{B}$$

$$= \frac{1}{c\mu_0} \underline{E} \times (\underline{n} \times \underline{E})$$

$$= \underbrace{c\epsilon_0 E^2}_{w} \underline{n}$$