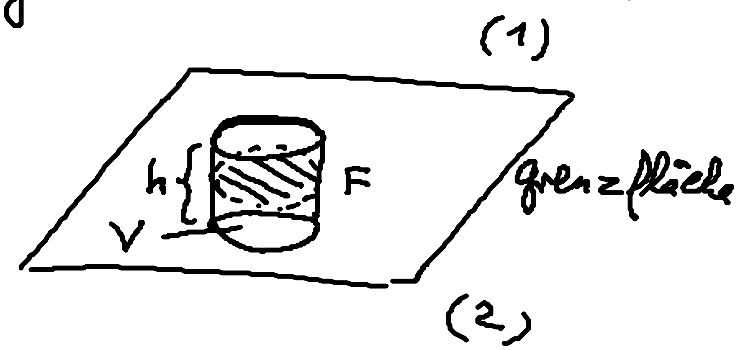
5.4 Grenzbedingungen für Felder

Frage: Wie verhalten sich $\underline{E}, \underline{D}, \underline{H}, \underline{B}$ an Grenzflächen zwischen verschiedenen Materialien?

Integration der Maxwell-Gln. über Volumen V:



$$(1)\quad \int_V d^3r\, \underline{\nabla} \times \underline{E} = -\int_V d^3r\, \dot{\underline{B}}$$

$$(2)\quad \int_V d^3r\, \underline{\nabla} \times \underline{H} = \int_V d^3r\, (\underline{j} + \dot{\underline{D}})$$

$$(3)\quad \int_V d^3r\, \underline{\nabla} \cdot \underline{B} = 0 \overset{\text{Gauß}}{=} \oint_{\partial V} d\underline{f} \cdot \underline{B}$$

$$(4)\quad \int_V d^3r\, \underline{\nabla} \cdot \underline{D} = \int_V d^3r\, \varrho \overset{\text{Gauß}}{=} \oint_{\partial V} d\underline{f} \cdot \underline{D}$$

Normalkomponenten

$\underline{h \to 0}$ $\quad (3) \Rightarrow \lim\limits_{h \to 0} \oint_{\partial V} d\underline{f} \cdot \underline{B} = \int_F d\underline{f} (\underline{B}^{(1)} - \underline{B}^{(2)})$

$$= \int_F df\, \underline{n} (\underline{B}^{(1)} - \underline{B}^{(2)})$$

$$(4) \Rightarrow \lim_{h \to 0} \oint_{\partial V} d\underline{f} \cdot \underline{D} = \int_F df\, \underline{n} (\underline{D}^{(1)} - \underline{D}^{(2)})$$

Annahme: freie Flächenladungsdichte σ

$$\varrho(\underline{r}, t) = \sigma(x, y, t)\, \delta(z) \qquad \underline{e}_z = \underline{n}$$

$$\Rightarrow \lim_{h\to 0} \int_V d^3r\, \rho = \int_F df\, \underline{\sigma}$$

bel. Fläche F $\Rightarrow$
$$\boxed{\begin{aligned} \underline{n}\,(\underline{B}^{(1)} - \underline{B}^{(2)}) &= 0 \\ \underline{n}\,(\underline{D}^{(1)} - \underline{D}^{(2)}) &= \sigma \end{aligned}}$$

<u>Tangentialkomponenten</u>

Verallg. Gauß'scher Satz:

$$(1) \Longrightarrow \oint_{\partial V} d\underline{f} \times \underline{E} = -\int_V d^3r\, \dot{\underline{B}}$$

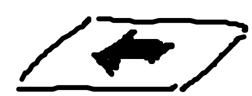
$$(2) \Longrightarrow \oint_{\partial V} d\underline{f} \times \underline{H} = \int_V d^3r\, (\underline{j} + \dot{\underline{D}})$$

$\underline{h \to 0}$ $\quad (1) \Rightarrow \lim_{h\to 0} \oint_{\partial V} d\underline{f} \times \underline{E} = \int_F df\, \underline{n} \times (\underline{E}^{(1)} - \underline{E}^{(2)})$ Tangentialkomponente

$$(2) \Rightarrow \lim_{h\to 0} \oint_{\partial V} d\underline{f} \times \underline{H} = \int_F df\, \underline{n} \times (H^{(1)} - H^{(2)})$$

freie Flächenstromdichte $\underline{g}$:

$$\underline{j}(\underline{r}, t) = \underline{g}(x, y, t)\, \delta(z)$$



$$\Rightarrow \lim_{h\to 0} \int_V d^3r\, \underline{j} = \int_F df\, \underline{g}$$

$\underline{B}, \underline{D}$ und $\dot{\underline{B}}, \dot{\underline{D}}$ beschränkt $\Rightarrow \lim_{h\to 0} \int_V d^3r\, \dot{\underline{B}} = 0$

$$\lim \int d^3r\, \dot{\underline{D}} = 0$$

für bel. F $\Longrightarrow$ $\quad h \to 0$ ✓

(1) $\int_F df\, \underline{n} \times (\underline{E}^{(1)} - \underline{E}^{(2)}) = 0$

(2) $\int_F df\, \underline{n} \times (\underline{H}^{(1)} - \underline{H}^{(2)}) = \int_F df\, \underline{g}$

$$\boxed{\begin{aligned} \underline{n} \times (\underline{E}^{(1)} - \underline{E}^{(2)}) &= 0 \\ \underline{n} \times (\underline{H}^{(1)} - \underline{H}^{(2)}) &= \underline{g} \end{aligned}}$$

Zusammenfassung : $\delta \underline{E} := \underline{E}^{(1)} - \underline{E}^{(2)}$

Maxwell-Gln.	Grenzbed.	
$\underline{\nabla} \times \underline{E} = -\dot{\underline{B}}$	$\underline{n} \times \delta\underline{E} = 0$	Tangentialkomp. v. $\underline{E}$ stetig
$\underline{\nabla} \cdot \underline{D} = \rho$	$\underline{n} \cdot \delta\underline{D} = \sigma$	Normalkomp. v. $\underline{D}$ springen
$\underline{\nabla} \times \underline{H} = \underline{j} + \dot{\underline{D}}$	$\underline{n} \times \delta\underline{H} = \underline{g}$	Tangentialkomp. v. $\underline{H}$ springen
$\underline{\nabla} \cdot \underline{B} = 0$	$\underline{n} \cdot \delta\underline{B} = 0$	Normalkomp. v. $\underline{B}$ stetig

5.6 Wellenausbreitung in Materie

$\underline{D} = \epsilon_0 \epsilon\, \underline{E} \qquad (\epsilon > 1)$

$\underline{B} = \mu_0 \mu\, \underline{H} \qquad (\text{i.a. } \mu \simeq 1)$

$\underline{j} = \sigma \underline{E} \qquad$ (Ohm'sches Gesetz)

a) Wellen in leitenden Medien ohne Dispersion

(d.h. ϵ, μ, σ unabh. v. ω)

$\rho = 0$

$$\underline{\nabla} \times \underline{E} + \underline{B} = 0$$

$$\underline{\nabla} \times \underline{B} - \mu_0 \mu \, \epsilon_0 \epsilon \, \dot{\underline{E}} \overset{!}{=} \mu_0 \mu \, \underline{j} = \mu_0 \mu \, \sigma \, \underline{E} \qquad \left| \begin{array}{l} \underline{\nabla} \cdot \underline{E} = 0 \\ \underline{\nabla} \cdot \underline{B} = 0 \end{array} \right.$$

$$\Longrightarrow \underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = \underline{\nabla} (\overbrace{\underline{\nabla} \cdot \underline{E}}^{0}) - \Delta \underline{E}$$

$$= - \underline{\nabla} \times \dot{\underline{B}} = - \mu_0 \mu \, \epsilon_0 \epsilon \, \ddot{\underline{E}} - \mu_0 \mu \, \sigma \, \dot{\underline{E}}$$

$$\boxed{\Delta \underline{E} - \frac{1}{c_M^2} \left(\ddot{\underline{E}} + \frac{\sigma}{\epsilon_0 \epsilon} \dot{\underline{E}} \right) = 0}$$

<u>gedämpfte</u> Wellengl. $\quad c_M := \frac{1}{\sqrt{\mu_0 \mu \, \epsilon_0 \epsilon}} = \frac{c}{\sqrt{\mu \epsilon}}$

(1-dim. <u>Telegraphengl.</u>, beschreibt Drahtwellenausbreitung)

<u>Harmon. ebene Welle</u>

$$\underline{E}(\underline{r}, t) = \underline{E}_0 \, e^{i(\underline{k} \cdot \underline{r} - \omega t)}$$

$$\Longrightarrow \boxed{\underline{k}^2 = \frac{\epsilon \mu}{c^2} \underline{\omega}^2 \left(1 + i \frac{1}{\omega \tau} \right)}$$ Dispersionsrelation

$\tau := \frac{\epsilon_0 \epsilon}{\sigma}$ <u>dielektr. Relaxationszeit</u>

Wellenvektor $k \in \mathbb{C}$ (wegen Dämpfung)

Setze $\boxed{k = \frac{\omega}{c}\tilde{n} = \frac{\omega}{c}(n + i\gamma)}$ c Vakuumlichtgesch

$\tilde{n} = n + i\gamma$ komplexer Brechungsindex

$\Rightarrow k^2 = \frac{\omega^2}{c^2}(n^2 - \gamma^2 + 2in\gamma) \overset{!}{=} \frac{\omega^2}{c^2}\epsilon\underline{\mu}\left(1 + \frac{i}{\omega\tau}\right)$

$$\left.\begin{array}{ll} \text{Re} & \boxed{n^2 - \gamma^2 = \epsilon\mu} \\ \text{Im} & \boxed{n\gamma = \frac{\epsilon\mu}{2\omega\tau}} \end{array}\right\} \Rightarrow n, \gamma \quad (*)$$

oBdA $\underline{k} \parallel x_3$: $\underline{E}(x_3, t) = \underline{E}_0\, e^{-\frac{x_3}{d}} e^{-i\omega(t - \frac{n}{c}x_3)}$

gedämpfte Welle mit
Phasengeschw. $\frac{c}{n}$
und Extinktionskoff. $d := \frac{c}{\omega\gamma}$

Isolator ($\sigma = 0$) : $\tau \to \infty \overset{(*)}{\Rightarrow} \gamma = 0$

ungedämpft

reeller Brech. index $\boxed{n = \sqrt{\epsilon\mu} \approx \sqrt{\epsilon} > 1}$

Phasengeschw. $\frac{c}{n} < c$

Metall (σ groß) : $\tau = \frac{\epsilon_0 c}{\sigma} \ll \frac{1}{\omega}$ (für alle Frequ. bis UV)

$\Rightarrow k^2 \equiv \frac{\omega^2}{c^2}(n^2 - \gamma^2 + 2in\gamma) \approx \frac{\omega^2}{c^2}\epsilon\mu\frac{i}{\omega\tau}$

$\overset{(*)}{\Longrightarrow}$ $n^2 - \gamma^2 \approx 0$

$n\gamma \approx n^2 \approx \gamma^2 \approx \frac{\epsilon\mu}{2\omega\tau} \Rightarrow \boxed{n = \gamma = \sqrt{\frac{\epsilon\mu}{2\omega\tau}}}$

Ext.koeff $d = \frac{c}{\omega\gamma}$ ~ cm für 100 Hz

$= c\sqrt{\frac{2\tau}{\epsilon\mu\omega}}$ ~ 10 μm für 100 MHz

hochfrequente Wellen dringen nicht in Metall ein!

b) <u>Diel. Dispersion</u> (Ann. $\mu = 1$)

zeitl. Dispersion $\hat{\chi}(\omega)$

$\epsilon = 1 + \chi$

$$\boxed{\hat{\underline{P}}(\omega) = \epsilon_0 \hat{\chi}(\omega)\, \hat{\underline{E}}(\omega)}$$

$$\hat{\chi}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, \chi(t)\, e^{i\omega t}$$ (dynam. el. Suszeptib.)

$$\underline{P}(\underline{r}, t) = \frac{\epsilon_0}{\sqrt{2\pi}} \int_{-\infty}^{t} dt'\, \chi(t - t')\, \underline{E}(\underline{r}, t')$$

Gedächtniseffekt?

i.a. $\hat{\chi}(\omega) \in \mathbb{C}$ <u>komplexe diel. Fkt.</u>

$$\epsilon(\omega) = 1 + \hat{\chi}(\omega) = \epsilon'(\omega) + i\epsilon''(\omega)$$

<u>monochromat. ebene Welle</u> $\underline{E}(\underline{r}, t) = \underline{E}_0 e^{i(\underline{k}\cdot\underline{r} - \omega t)}$

$$\Rightarrow \boxed{k^2 = \underline{\epsilon(\omega)} \frac{\omega^2}{c^2}\left(1 + i\frac{1}{\omega\tau}\right)} \quad (**)$$

<u>Isolator</u> (dispersives Diel.) $\boxed{k^2 \approx \epsilon(\omega)\frac{\omega^2}{c^2}}$

$$\tilde{n}(\omega) = n(\omega) + i\kappa(\omega)$$

$$\tilde{n}(\omega)^2 = \epsilon(\omega) \equiv \epsilon' + i\epsilon''$$

$$(**)\quad \left.\begin{aligned} \epsilon'(\omega) &= n^2 - \kappa^2 \\ \epsilon''(\omega) &= 2n\kappa \end{aligned}\right\} \Rightarrow \boxed{\left.\begin{matrix} \kappa \\ n \end{matrix}\right\} = \frac{1}{\sqrt{2}}\left(\sqrt{\epsilon'^2 + \epsilon''^2} \mp \epsilon'\right)^{1/2}}$$

Abs. koeff. κ

rell. Brech. index n