b) Der Formalismus der linearen Antwort I
(Methode der Greenschen Fkt.)

- dynam. Eigenschaften eines Systems?
  äußerer Stimulus/erregende Kraft $\longrightarrow$ Reaktion/Antwort

  Bsp: $A_\omega = \chi(\omega)\frac{F_\omega}{m}$ ... „linear"

- <u>Ziel</u>: „allgemeinste <u>lineare</u> Antwort"

- Bsp: harm. Oszillator: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \hat{F}(t) = \frac{F(t)}{m}$ (5.40)

  beliebige Kraft $\hat{F}(t)$ $\xrightarrow[\text{Antwort}]{\text{„allg. lineare}}$ $x_{part}(t) = \underbrace{\int_{-\infty}^{t} G(t-t')\hat{F}(t')dt'}_{\text{„Faltung"}}$ (5.41)

  (bzw: $I(t) = \int_{-\infty}^{t}\hat{F}(t')dt'$

  ... Kraftstoß)

  (i) $\int_{-\infty}^{t}$ ... Kausalitätsprinzip („Wirkung nach der Ursache")

  (ii) $G(t-t')$ ... „Greensche Fkt"

- Führe ein:
$$G_c(t-t') = \begin{cases} G(t-t') & , t>t' \\ 0 & , t<t' \end{cases}$$
... <u>kausale Greensche Fkt.</u>

(5.41) $\longrightarrow$ $\boxed{x_{part}(t) = \int_{-\infty}^{\infty} G_c(t-t')\hat{F}(t')dt'}$ (5.42)

· Bestimmungsgleichung für $G_c(t-t')$:

(5.42) in (5.40) $\longrightarrow$ $$\tilde{F}(t) = \int_{-\infty}^{\infty} \underbrace{\left[\frac{d^2}{dt^2} + 2\gamma \frac{d}{dt} + \omega_0^2\right]}_{\hat{L}(t)} G_c(t-t')\, \tilde{F}(t')\, dt' \quad (5.43)$$

Differentialoperator

$\longrightarrow$ formal: $\underbrace{\hat{L}(t) G_c}_{= \text{"}1\text{"}} \tilde{F} = \tilde{F}$

... Eins-Operator

hier „Zeitraum": Führe ein „δ-Funktion" (Distribution)

so daß: $$\boxed{\tilde{F}(t) = \int_{-\infty}^{\infty} \underbrace{\delta(t-t')}_{} \tilde{F}(t')\, dt'} \quad (5.44)$$

1-Operator im Zeitraum:
$\delta(t-t')$ stark gepeakt
bei $t=t'$

vgl. mit (5.43) $\longrightarrow$ $$\boxed{\hat{L}(t)\, G_c(t-t') = \delta(t-t')} \quad (5.45)$$

c) Die Diracsche „δ-Funktion"

· Grundeigenschaft: $$\boxed{\int_{-\infty}^{\infty} \delta(x-x')\, f(x')\, dx' = f(x)} \quad (5.46)$$

$\delta(x-x')$ ... 1-Operator im Raum der Fktn. $f(x)$

[vgl. $a_i = \sum_j \delta_{ij} a_j$, $\underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$

hier: $\sum_j \delta_{ij} \longrightarrow \int \delta(x-x') \dots dx'$]

- „δ-Fkt." als Grenzfall __stetiger__ Fktn.:

__Satz:__

$$\boxed{\begin{array}{l} \delta(x-x') = \text{“}\lim_{\varepsilon \to 0} \delta_\varepsilon(x-x')\text{”} \\ \text{wobei 1. } \delta_\varepsilon(x-x') \underset{\xi = \frac{x-x'}{\varepsilon}}{=} \frac{1}{\varepsilon} \underbrace{g(\xi)}_{\text{„gutartig“}} \\ \quad 2. \int_{-\infty}^{\infty} \delta_\varepsilon(x-x') \underbrace{d(x-x')}_{\varepsilon\, d\xi} = \int_{-\infty}^{\infty} g(\xi)\, d\xi = 1 \\ \quad 3. \text{ “…”: in (5.46): erst } \int \dots \text{, dann } \lim_{\varepsilon \to 0} \end{array}} \quad (5.47)$$

(5.47) → (5.46) ? __Bew:__

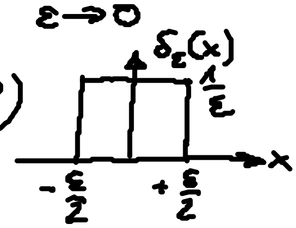
$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \underbrace{\delta_\varepsilon(x-x')}_{\frac{1}{\varepsilon} g(\xi)} f(x')\, dx' \underset{\substack{\frac{x-x'}{\varepsilon} = \xi \\ \varepsilon\, d\xi = -dx'}}{=\!=} \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \underbrace{g(\xi) f(x - \varepsilon\xi)}_{\text{“ordentlich”}} d\xi$$

$\longrightarrow \int \lim_{\varepsilon \to 0} \dots$

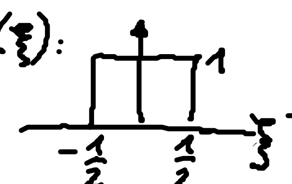
$$= f(x) \underbrace{\int_{-\infty}^{\infty} g(\xi)\, d\xi}_{1} = f(x)$$

⟶ i.f. Rechne mit $\delta(x-x')$ ohne $\lim_{\varepsilon \to 0}$

- Bsp: (i) $\delta_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon} & \text{für } |x| \le \frac{\varepsilon}{2} \\ 0 & \text{" } |x| > \frac{\varepsilon}{2} \end{cases}$ (5.48)

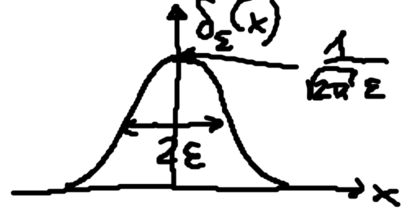


[$g(\xi)$:



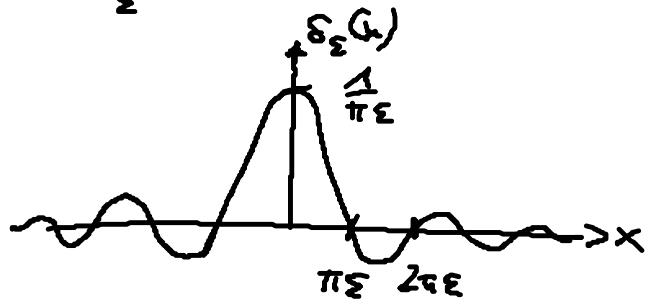
]

(ii) $\delta_\varepsilon(x) = \frac{1}{\sqrt{2\pi}\,\varepsilon} e^{-x^2/2\varepsilon^2}$ (5.49) … Gaußfkt. (Glockenfkt.)



(iii) $\delta_\varepsilon(x) = \frac{1}{\pi\varepsilon}\,\frac{1}{1+\left(\frac{x}{\varepsilon}\right)^2}$ ... Lorentz-Kurve (5.50)

(iv) $\delta_\varepsilon(x) = \frac{1}{\pi\varepsilon}\,\frac{\sin x/\varepsilon}{x/\varepsilon} \underset{\text{o.B.}}{=} \frac{1}{2\pi}\int_{-\frac{1}{\varepsilon}}^{1/\varepsilon} dq\, e^{iqx}$ (5.51) ... Spaltbeugungs-fkt. der Optik



· <u>Eigenschaften der δ-Funktion</u>

(i) $\delta(x) = \begin{cases} 0 & , x \neq 0 \\ \text{"}\infty\text{"} & , x = 0 \end{cases}$ (5.52)

(ii) $\int_a^b \delta(x-x')\,f(x')\,dx' = \begin{cases} f(x) & , a<x<b \\ 0 & , \text{sonst} \end{cases}$ (5.53)

(iii) $\delta(ax) = \frac{1}{|a|}\,\delta(x)$ (5.54)

$\delta(-x) = \delta(x)$ ... gerade Fkt (5.55)

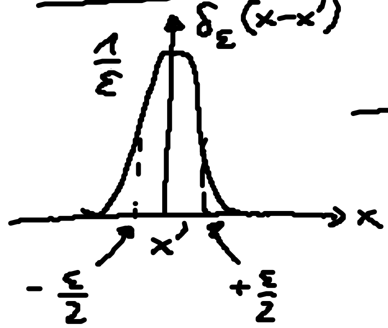
(iv) $\delta(x) \underset{(5.51)}{=} \int_{-\infty}^{\infty} e^{\pm iqx}\,\frac{dq}{2\pi}$ (5.56)

... Fourier-Darstellung

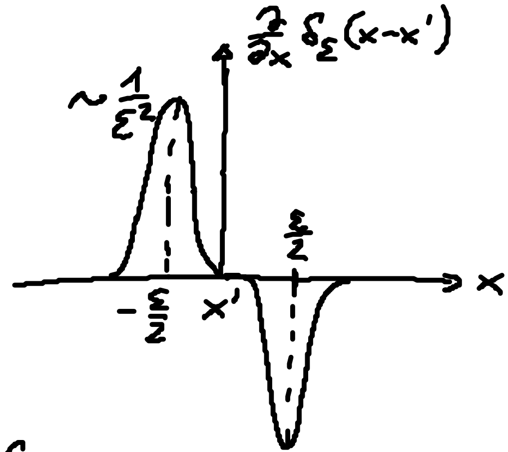
(v) $\int_{-\infty}^{\infty} \frac{\partial}{\partial x}\,\delta(x-x')\,f(x')\,dx' = -\int \left[\frac{\partial}{\partial x'}\,\delta(x-x')\right] f(x')\,dx'$

$= f'(x)$ (5.57)

anschaulich



$\longrightarrow$



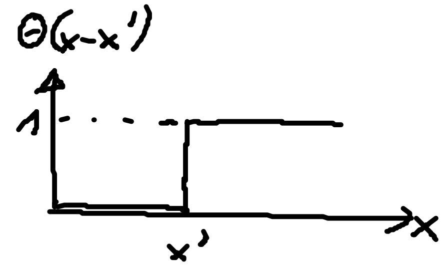
$$\int \frac{\partial}{\partial x}\delta_\varepsilon(x-x')\,f(x')\,dx' \sim \frac{1}{\varepsilon}\int\left[\delta_\varepsilon\left(x-x'+\frac{\varepsilon}{2}\right)-\delta_\varepsilon\left(x-x'-\frac{\varepsilon}{2}\right)\right]f(x')\,dx'$$

$$\approx \underbrace{\frac{1}{\varepsilon}\left[f\left(x+\frac{\varepsilon}{2}\right)-f\left(x-\frac{\varepsilon}{2}\right)\right]}_{\text{zentrierte Abl.}} \longrightarrow f'(x)\,,\quad \varepsilon\to 0$$

Beweis: (iii)/(v) : Übungen

- Stufenfkt.

(i) $$\boxed{\Theta(x-x') = \begin{cases} 1, & x-x'>0 \\ 0, & x-x'<0 \end{cases}} \quad (5.58)$$



(ii) $$\Theta(x) \underset{(5.53)}{=} \int_{-\infty}^{x} \delta(x')\,dx' \quad (5.59)$$

$$\longrightarrow \quad \boxed{\delta(x) = \frac{d}{dx}\Theta(x)} \quad (5.60)$$

d) Formalismus der linearen Antwort II

- Es galt: $\hat{L}(t)\,G_c(t-t') = \underbrace{\delta(t-t')}_{\delta\text{-Kraft}}$, $\hat{L}(t) = \frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2$ (5.61)

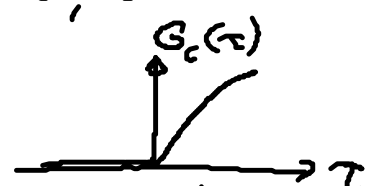
$\quad$, $\int \delta(t-t')\,dt = 1$ … δ-Kraftstoß

- Bestimmung der Greensche Fkt. aus Stetigkeits- und Sprungbed.:

(i) Kausalität: $\boxed{G_c(\tau) = 0\,,\quad \tau = t-t' < 0}$

(ii) Stetigkeit: $G_c(\varepsilon) - \underbrace{G_c(-\varepsilon)}_{=0} = 0, \quad \varepsilon \to 0$

$$\rightarrow \boxed{G_c(0) = 0}$$



Knick [s. (iii)]