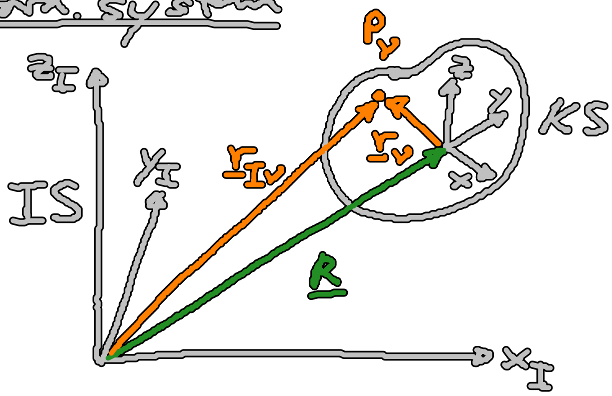
10. Starrer Körper

10.1. Kinematik

a) Koord. system



b) Eulerscher Satz:

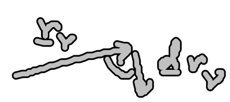
Jede Bewegung des starren Körpers läßt sich zu jedem Zeitpkt. zerlegen in eine Translation des Aufpunktes $\underline{R}(t)$ und eine Rotation um eine momentane Drehachse $\underline{\omega}(t)$ durch den Aufpkt.

$$\dot{\underline{r}}_{I\nu}(t) = \dot{\underline{R}}(t) + \underline{\omega}(t) \times \underline{r}_\nu(t) \quad \forall P_\nu \qquad (10.4)$$

Beweis: (i) $d\underline{r}_{I\nu}(t) \overset{(10.3)}{=} d\underline{R}(t) + d\underline{r}_\nu(t)$ (*)

$\underline{r}_{I\nu} = \underline{R} + \underline{r}_\nu$ (10.3)

starrer Körper: $|\underline{r}_\nu| = \text{konst.} \; \forall t \longrightarrow d\underline{r}_\nu \perp \underline{r}_\nu$



$\xrightarrow[(7.5)]{} d\underline{r}_\nu = d\varphi\, \underline{s}_\omega \times \underline{r}_\nu$

mit $\underline{\omega}_\nu = \frac{d\varphi}{dt}\underline{s}_\omega$ & $\frac{(*)}{dt}$: $\dot{\underline{r}}_{I\nu}(t) = \dot{\underline{R}}(t) + \underline{\omega}_\nu \times \underline{r}_\nu$

(ii) $\underline{\omega} = \underline{\omega}_\nu = \underline{\omega}_\mu$? $\qquad \dot{\underline{r}}_{I\mu}(t) = \dot{\underline{R}}(t) + \underline{\omega}_\mu \times \underline{r}_\mu$

Rel. geschw.: $\dot{\underline{r}}_{I\nu} - \dot{\underline{r}}_{I\mu} \overset{(10.3)}{=} \dot{\underline{r}}_\nu - \dot{\underline{r}}_\mu$

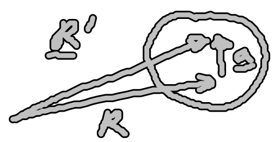
$$= \underline{\omega}_\nu \times \underline{r}_\nu - \underline{\omega}_\mu \times \underline{r}_\mu \perp \underline{r}_\nu - \underline{r}_\mu, \quad \forall \nu\mu$$

nur mit $\underline{\omega}_\nu = \underline{\omega}_\mu = \underline{\omega}$

$$\longrightarrow \dot{\underline{r}}_\nu - \dot{\underline{r}}_\mu = \underline{\omega} \times (\underline{r}_\nu - \underline{r}_\mu) \quad \text{qed}$$

• Satz: $\boxed{\underline{\omega} \text{ ist unabhängig von } \underline{R}}$

Beweis: Aufpunkte: $\underline{R}, \underline{R}' = \underline{R} + \underline{a} \longrightarrow \underline{r}_{I\nu} = \underline{R} + \underline{r}_\nu = \underline{R}' + \underbrace{\underline{r}'_\nu}_{\underline{r}_\nu - \underline{a}}$



(10.4) $\dot{\underline{r}}_{I\nu} = \dot{\underline{R}} + \underline{\omega} \times \underline{r}_\nu = \dot{\underline{R}}' + \underline{\omega}' \times \underline{r}'_\nu$

$$= \dot{\underline{R}} + \underbrace{\dot{\underline{a}}}_{\underline{\omega} \times \underline{a}} + \underline{\omega}' \times (\underline{r}_\nu - \underline{a})$$

$$\longrightarrow (\underline{\omega} - \underline{\omega}') \times (\underline{r}_\nu - \underline{a}) = 0 \quad \forall \nu$$

$$\longrightarrow \underline{\omega} = \underline{\omega}' \quad \text{qed}$$

c) **Eulersche Winkel**:

• Lage des starren Körpers:

(i) $\underline{R} \longrightarrow$ 3 Koord.

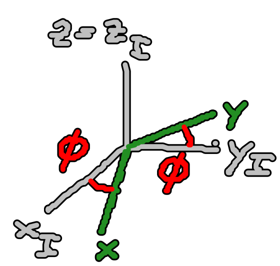
(ii) Orientierung von KS relativ zu IS

$\longrightarrow$ 3 Eulersche Winkel

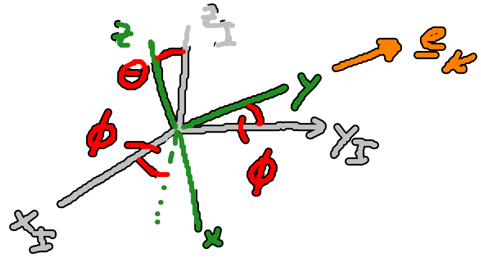
$\longrightarrow f = 6!$

• Schritte mit IS = KS:

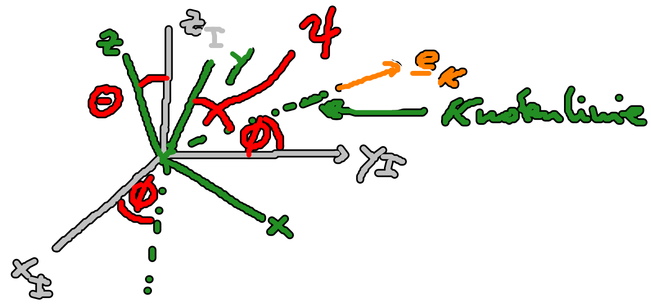
ϕ: Rot. um $z_I = z$-Achse:



Θ: Rot. um momentane y-Achse



ψ: Rot. um z-Achse



- von $\underline{\omega}$ zu ϕ, Θ, ψ : $\longrightarrow$ Folie

10.2 Dynamik des starren Körpers (I) & Trägheitstensor

Dynam.

a) Grundgl.

- starrer Körper $\equiv$ Spezialfall eines Vielteilchen-Systems
- innere Kräfte:

$$\underline{F}_{\nu\mu} = -\underline{F}_{\mu\nu} \longrightarrow \underline{F}^{(i)} = \sum_{\nu\mu} \underline{F}_{\nu\mu} = 0 \quad (8.6) \; \dots \text{„Münchhausen"}$$

$$\underline{F}_{\nu\mu} \parallel \underline{r}_{I\nu} - \underline{r}_{I\mu} \longrightarrow \underline{D}^{(i)} = \sum_{\nu\mu} \underline{r}_{I\nu} \times \underline{F}_{\nu\mu} = 0 \quad (8.7) \; \dots \text{„keine spontane Rotation"}$$

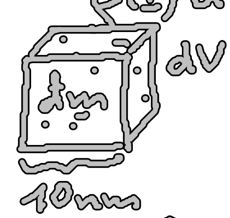
starrer Körper: (i) $\underline{F}^{(i)} = 0 = \underline{D}^{(i)}$

(ii) innere Kräfte $\longrightarrow$ "Starrheit"

- Kontinuumslimes:

diskrete Massenpkte $m_\nu \longrightarrow$ kont. Massenverteilung: $dm = \rho(\underline{r})\, d^3r$

Massendichte: $\rho(\underline{r}) = \dfrac{dm}{dV = d^3r}$



$\longrightarrow$ 10^4–10^6 Atome in dV

$$\sum_\nu m_\nu \dots \longrightarrow \int \underbrace{d^3r\, \rho(\underline{r})}_{dm} \dots$$

Bsp: (1) Gesamtmasse: $M = \sum_\nu m_\nu \longrightarrow \int_V d^3r\, \rho(r)$ (10.10)

(2) " impuls: $\underline{P} = \sum_\nu m_\nu \underline{\dot{r}}_{I\nu} \longrightarrow \int d^3 r_I\, \rho(r_I)\, \underline{v}(r_I)$ (10.11)

(3) " drehimpuls: $\underbrace{\underline{L}_I = \sum_\nu m_\nu r_{I\nu} \times \underline{\dot{r}}_{I\nu}}_{\text{im folgenden!!}} \longrightarrow \int d^3 r_I\, \rho(r_I)\, \underline{r}_I \times \underline{v}(r_I)$ (10.12)

- Bewegungsgln.

$$\underline{\dot{P}} = \sum_\nu \underline{F}_\nu^{(a)} = \underline{F}^{(a)} \quad (10.13) \quad [(8.8)]$$

$$\underline{\dot{L}}_I = \sum_\nu r_{I\nu} \times \underline{F}_\nu^{(a)} = \sum_\nu \underline{D}^{(a)} = \underline{D}^{(a)} \quad (10.14) \quad [(8.14)]$$

... 6 Gl., die die Dynamik des starren Körpers ($f=6$) eindeutig bestimmen.

- Gleichgewichtsbed.: $\left.\begin{array}{l}\underline{F}^{(a)} = 0\\ \underline{D}^{(a)} = 0\end{array}\right\} \left\{\begin{array}{l}\underline{\dot{P}} = 0\\ \underline{\dot{L}}_I = 0\end{array}\right. \longrightarrow$ Grundlage der Statik!

- im folgenden: 2 Fälle:

$$\boxed{\begin{array}{l}\text{(i) Aufpkt } \underline{R} = \text{Schwerpkt. } \underline{R}_S = \dfrac{\sum_\nu m_\nu \underline{r}_{I\nu}}{M}\\ \quad \text{Bsp: Erde, freifallender Kreisel}\\ \text{(ii) } \underline{R} \text{ mit } \underline{\dot{R}} = 0\\ \quad \text{Bsp: Kreisel mit festem Auflagepkt.}\end{array}} \quad (10.15)$$

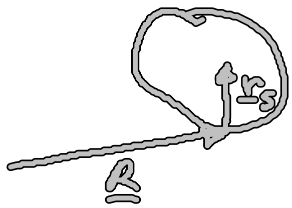
Folgerungen mit (10.4): $\underline{\dot{r}}_{I\nu}(t) = \underline{\dot{R}}(t) + \underline{\omega}(t) \times \underline{r}_\nu(t)$

b) **Impuls**

(i) $\underline{R} = \underline{R}_S$:
$$\underline{P} = \sum_\nu m_\nu \underline{\dot{r}}_{I\nu} \overset{(10.4)}{=} M \underline{\dot{R}}(t) + \underline{\omega}(t) \times \underbrace{\sum_\nu m_\nu \underline{r}_\nu(t)}_{=0 \text{ wegen } \underline{R} = \underline{R}_S}$$

$$\longrightarrow \boxed{\underline{P} = M \underline{\dot{R}}_S} \qquad (10.16)$$

(ii) $\underline{\dot{R}} = 0$
$$\boxed{\underline{P} = \underline{\omega} \times \sum_\nu m_\nu \underline{r}_\nu = \underline{\omega} \times M \underline{r}_S} \qquad (10.17)$$
$$= M \underline{\dot{r}}_S$$



c) **Drehimpuls**

$$\underline{L}_I = \sum_\nu m_\nu \underline{r}_{I\nu} \times \underline{\dot{r}}_{I\nu}$$
$$\underline{r}_{I\nu} = \underline{R} + \underline{r}_\nu \qquad \underline{\dot{r}}_{I\nu} = \underline{\dot{R}} + \underline{\omega} \times \underline{r}_\nu$$

(i) $\underline{R} = \underline{R}_S$:

$\sum_\nu m_\nu \underline{r}_\nu = 0 \longrightarrow$

$$\boxed{\begin{aligned} &\underline{L}_I = \underline{L}_S + \underline{L} && (10.18)\\ &\text{mit } \underline{L}_S = M \underline{R}_S \times \underline{\dot{R}}_S && (10.19)\\ &\underline{L} = \sum_\nu m_\nu \underline{r}_\nu \times (\underline{\omega} \times \underline{r}_\nu) && (10.20) \end{aligned}}$$

$\underline{L}_I$ ...	Drehimpuls		bzgl. Ursprung von IS
$= \underline{L}_S$ ...	"	des Schwerpkts.	"
$+ \underline{L}$ ...	"	des starren Körpers bzgl. Aufpkt.	

(ii) $\underline{\dot{R}} = 0$: $\longrightarrow$
$$\underline{L}_I = \underbrace{\underline{R} \times (\underline{\omega} \times \underbrace{\sum_\nu m_\nu \underline{r}_\nu}_{M \underline{r}_S})}_{\underline{P} \text{ [s. (10.17)]}} + \underbrace{\underline{L}}_{(10.20)}$$

$$\longrightarrow \boxed{\underline{L}_I = \underline{L}_S + \underline{L} = \underline{R} \times \underline{P} + \underline{L}} \qquad (10.21)$$

wie (10.18)–(10.20)

d) Trägheitstensor: Umschreibung von (10.20)

• Hilfsformel: $\underline{a} \times (\underline{b} \times \underline{a}) = |\underline{a}|^2 \underline{b} - \underline{a}(\underline{a}\cdot\underline{b})$

$\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a}\cdot\underline{c}) - \underline{c}(\underline{a}\cdot\underline{b})$

$$= |\underline{a}|^2 \underline{b} - (\underline{a}\otimes\underline{a})\,\underline{b}$$

$$= \left(|\underline{a}|^2 \underline{\underline{1}} - \underline{a}\otimes\underline{a}\right)\underline{b} \qquad (10.22)$$

[NB: $\underline{u}\otimes\underline{v}$ ... dyadisches Produkt von $\underline{u}, \underline{v}$; spezieller Tensor 2.St.

$$(\underline{u}\otimes\underline{v})\,\underline{b} = \underline{u}\;\underline{v}\cdot\underline{b} \qquad (10.23)$$]

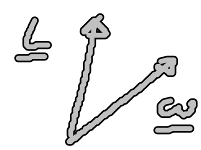
• Drehimpuls $\underline{L}$: (10.20) mit (10.22) $(\underline{\omega} = \underline{b},\ \underline{r}_\nu = \underline{a})$ (10.24)

$$\boxed{\underline{L} = \underline{\underline{\Theta}}\,\underline{\omega} \quad \text{mit} \quad \underline{\underline{\Theta}}(t) = \sum_\nu m_\nu \left[|\underline{r}_\nu|^2 \underline{\underline{1}} - \underline{r}_\nu(t)\otimes\underline{r}_\nu(t)\right]}$$

... Trägheitstensor in koord. unabh. Form

(i) $\underline{\underline{\Theta}}$ ... Eigenschaft des starren Körpers

(ii) lin. Abb.



• Komponentendarstellung bzgl. ONB $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

$$\underline{e}_i\cdot\ \Big|\ \underline{L} = \underline{\underline{\Theta}}\,\underbrace{\underline{\omega}}_{\omega_j \underline{e}_j} \longrightarrow L_i = \underline{e}_i \cdot \underline{\underline{\Theta}}\,\underline{e}_j\,\omega_j$$

(10.25)

$$\boxed{L_i = \Theta_{ij}\,\omega_j \quad \text{mit} \quad \Theta_{ij} = \underline{e}_i \cdot \underline{\underline{\Theta}}\,\underline{e}_j = \sum_\nu m_\nu \left[|\underline{r}_\nu|^2 \delta_{ij} - x_{\nu i}\, x_{\nu j}\right]}$$