12.5 Lagrangesche Gleichungen 2. Art

- generalisierte Koord.: $q_1, \dots, q_f$, $f = 3N - Z$

$$\underline{r}_i = \underline{r}_i(q_1, \dots, q_f, t) \quad (12.4)$$

- Ziel: Beschreibe Dynamik der $q_1 \cdots q_f \xrightarrow{(12.4)} \underline{r}_i(t)$

  kein Zugang zu Zwangskräften

- Hilfsformeln: (i) $\dot{\underline{r}}_i \overset{(12.4)}{=} \sum_j \frac{\partial \underline{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \underline{r}_i}{\partial t} \quad (12.22)$

  Kettenregel

  (ii) $\delta \underline{r}_i = \sum_j \frac{\partial \underline{r}_i}{\partial q_j} \delta q_j$, $\delta t = 0 \quad (12.23)$

- d'Alembert: $\sum_i \left(\underline{F}_i^{(t)} - m_i \ddot{\underline{r}}_i \right) \cdot \delta \underline{r}_i = 0 \quad (12.5)$

(i) $$\sum_i \underline{F}_i^{(t)} \cdot \delta \underline{r}_i \overset{(12.23)}{=} \sum_j \underbrace{\sum_i \underline{F}_i^{(t)} \cdot \frac{\partial \underline{r}_i}{\partial q_j}}_{Q_j} \delta q_j = \sum_j Q_j \, \delta q_j \quad (12.24)$$

$$\boxed{Q_j = \sum_i \underline{F}_i^{(t)} \cdot \frac{\partial \underline{r}_i}{\partial q_j}} \quad (12.25) \qquad Q_j \dots \text{generalisierte Kräfte}$$

(ii) $$\sum_i m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i \underset{(12.23)}{=} \sum_j \sum_i \underbrace{m_i \ddot{\underline{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j}}_{} \delta q_j \qquad \frac{d}{dt}(\dots) = \sum_j \frac{\partial \dots}{\partial q_j} \dot{q}_j + \frac{\partial}{\partial t} \cdots$$

$$\frac{d}{dt}\left(m_i \dot{\underline{r}}_i \cdot \underbrace{\frac{\partial \underline{r}_i}{\partial q_j}}_{\overset{(12.22)}{=} \frac{\partial \dot{\underline{r}}_i}{\partial \dot{q}_j}} \right) - m_i \dot{\underline{r}}_i \cdot \underbrace{\frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_j} \right)}_{= \frac{\partial}{\partial q_j} \dot{\underline{r}}_i}$$

$$= \sum_j \sum_i \left\{ \frac{d}{dt} \left(\underbrace{m_i \dot{\underline{r}}_i \cdot \frac{\partial \dot{\underline{r}}_i}{\partial \dot{q}_j}}_{} \right) - \underbrace{m_i \dot{\underline{r}}_i \cdot \frac{\partial}{\partial q_j} \dot{\underline{r}}_i}_{} \right\} \delta q_j$$

$$= \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2} m_i \dot{\underline{r}}_i^2\right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2} m_i \dot{\underline{r}}_i^2\right)$$

mit $\boxed{T = \sum_i \frac{1}{2} m_i \dot{\underline{r}}_i^2}$

… kinetische Energie

$$\longrightarrow \sum_i m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \sum_j \left\{ \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j} T - \frac{\partial T}{\partial q_j} \right\} \delta q_j$$

(12.15): (i) − (ii) = 0

$$\sum_j \left\{ Q_j - \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \right) \right\} \underbrace{\delta q_j} = 0$$

alle unabhängig voneinander: holonome Bindungen!

$\delta q_k \neq 0, \quad \delta q_j = 0, \; j \neq k$

$$\boxed{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j} \quad (12.26) \quad , j = 1, \dots, f$$

… Lagrangesche Gln. 2. Art (im allgemeinen)

<u>Sonderfälle</u>

a) <u>konservative Systeme</u>: $\boxed{\text{wichtig!}}$

$$\boxed{\underline{F}_i^{(t)} = -\underline{\nabla}_i U} \quad (12.27) \quad \text{mit } U = U(\underline{r}_1, \dots, \underline{r}_N)$$

$$= U(\underline{r}_1(\dots q_j \dots, t) \dots \underline{r}_N(\dots))$$

damit $Q_j \overset{(12.25)}{=} \sum_i \underline{F}_i^{(t)} \cdot \frac{\partial \underline{r}_i}{\partial q_j} = \sum_i -\underline{\nabla}_i U \cdot \frac{\partial \underline{r}_i}{\partial q_j}$

$$\longrightarrow \boxed{Q_j = -\frac{\partial U}{\partial q_j}} \quad (12.28)$$

also: (12.26) $\longrightarrow \frac{d}{dt}\frac{\partial T - U}{\partial \dot{q}_j} - \frac{\partial (T-U)}{\partial q_j} = 0$

da $\frac{\partial U}{\partial \dot{q}_j} = 0$

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0} \quad j = 1, \dots f \quad (12.29)$$

... Lagrangesche Gln.
(" " 2. Art, im speziellen)

mit $\boxed{L = T - U}$ (12.30)

... Lagrange Funktion

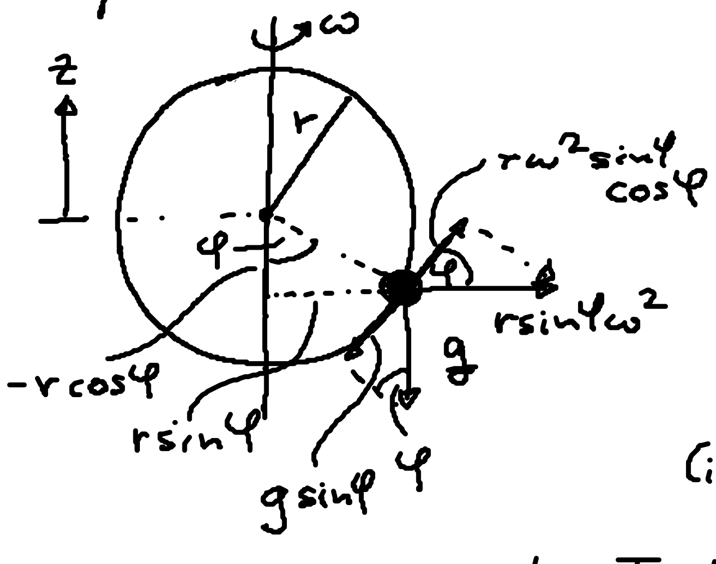
· Bsp. 1: Teilchen in 3D: $\underline{r} = (x_1, x_2, x_3) \longrightarrow q_j = x_j$, $j = 1,2,3$

$$\left.\begin{array}{l} T = \frac{1}{2} m \sum_j \dot{x}_j^2 \\ U = U(x_1, x_2, x_3) \end{array}\right\} L = T - U$$

$$\frac{\partial L}{\partial x_j} = -\frac{\partial U}{\partial x_j} = F_j \,, \quad \frac{\partial L}{\partial \dot{x}_j} = m\dot{x}_j$$

$$\xrightarrow{(12.29)} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} - \frac{\partial L}{\partial x_j} = 0 \longrightarrow m\ddot{x}_j = F_j \quad \checkmark$$

· Bsp 2: Perle im rotierenden Drahtring (ohne Reibung)



(i) generalisierte Koord.: φ

(ii) Geschw. komp.:

$$\underbrace{r\dot{\varphi}}_{\text{auf Ring}} \quad \& \quad \underbrace{\omega r \sin\varphi}_{\text{rot. Ring}}$$

$$\longrightarrow T = \frac{m}{2} r^2 (\dot{\varphi}^2 + \omega^2 \sin^2\varphi)$$

(iii) $U = mgz = -mg\underline{r\cos\varphi}$

$$\longrightarrow L = T - U$$
$$= \frac{m}{2} r^2 (\dot{\varphi}^2 + \omega^2 \sin^2\varphi) + mgr\cos\varphi$$

(iv) <u>Bewgl.</u>: $\frac{\partial L}{\partial \varphi} = mr^2\omega^2 \sin\varphi\cos\varphi - mgr\sin\varphi$

$$= mr(r\omega^2 \sin\varphi\cos\varphi - g\sin\varphi)$$

$$\frac{\partial L}{\partial \dot\varphi} = m r^2 \dot\varphi$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} - \frac{\partial L}{\partial \varphi} = mr\Big[\underbrace{r\ddot\varphi}_{\text{Beschl. von } m} - \big(\underbrace{r\omega^2 \sin\varphi\cos\varphi}_{\text{Zentrifugal-Beschl.}} - \underbrace{g\sin\varphi}_{\text{Erd-Beschl.}}\big)\Big] = 0$$

Komp. $\parallel$ Ring

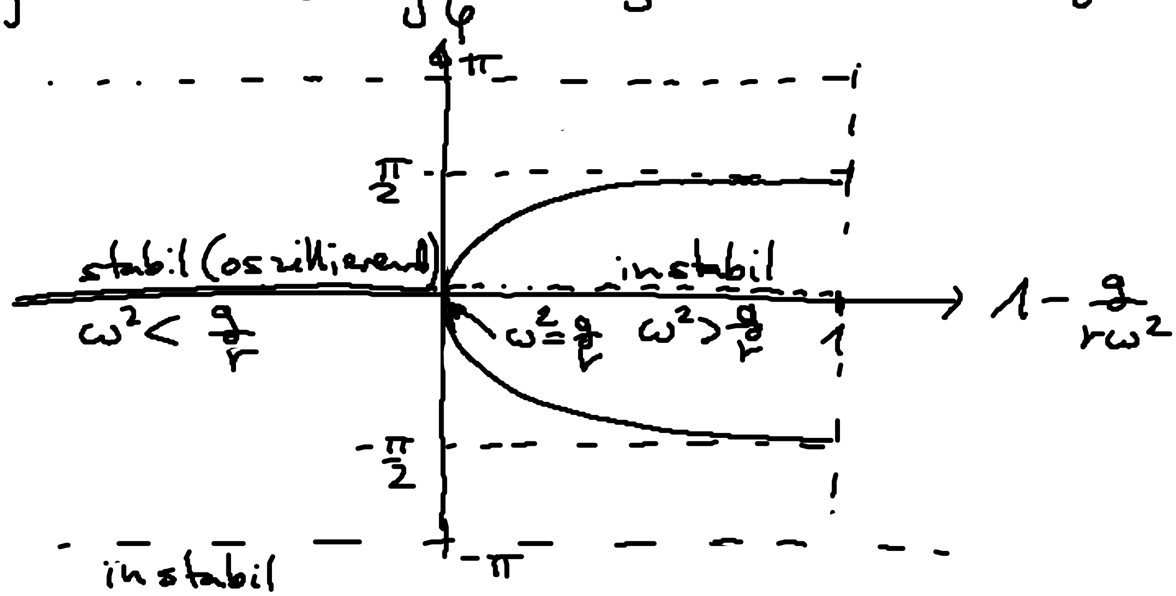
(v) Diskussion:

1. stationäre Lsg: $\dot\varphi = 0 \to \ddot\varphi = 0 \to \sin\varphi_1 = 0,\ \varphi_1 = 0, 180°$

   $\cos\varphi_2 = \frac{g}{r\omega^2}$

2. Verhalten gegen Störungen

3. Bifurkationsdiagramm für stationäre Lsg.



$$\boxed{\varphi_2 = \pm\sqrt{2}\sqrt{1 - \frac{g}{r\omega^2}}}$$

$\hat{=}$ Bifurkation der $\varphi = 0$ Lsg. in zwei gleichw. Lsg. (Symmetriebrechung!)