14.4. Liouvillesche Satz

- $$\underline{X}(t) = \begin{pmatrix} q_1 \\ \vdots \\ q_{3N} \\ p_1 \\ \vdots \\ p_{3N} \end{pmatrix} \quad \text{... Punkt im Phasenraum}$$

- Dichte $\varrho(\underline{X},t)$ im Phasenraum

  $\varrho(\underline{X},t)\, d^{6N}X$ ... Wahrscheinlichkeit, System im Zustand/Gebiet $[\underline{X}, \underline{X}+d\underline{X}]$ anzutreffen

  $\int_V \varrho \, d^{6N}X = 1$ ... Normierung (14.31)

- Kontinuitätsgl.:
$$\boxed{\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \dot{\underline{X}}) = 0} \quad (14.36)$$

  mit
$$\operatorname{div} = \underline{\nabla}_X \cdot = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_{6N}} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial q_1} \\ \vdots \\ \frac{\partial}{\partial p_{3N}} \end{pmatrix}$$

- Liouvillescher Satz:

  Herleitung:
$$\operatorname{div}(\varrho \dot{\underline{X}}) = \sum_j \left[\frac{\partial}{\partial q_j}(\varrho \dot{q}_j) + \frac{\partial}{\partial p_j}(\varrho \dot{p}_j)\right]$$

$$\underline{X} = \begin{pmatrix} q_1 \\ \vdots \\ q_{3N} \\ p_1 \\ \vdots \\ p_{3N} \end{pmatrix}$$

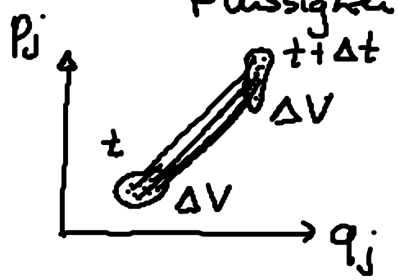
$$= \sum_j \Bigg[\frac{\partial}{\partial q_j}\varrho \underbrace{\dot{q}_j}_{=\frac{\partial H}{\partial p_j}} + \frac{\partial}{\partial p_j}\varrho \underbrace{\dot{p}_j}_{=-\frac{\partial H}{\partial q_j}} + \varrho \underbrace{\Bigg(\frac{\partial}{\partial q_j} \underbrace{\dot{q}_j}_{\frac{\partial H}{\partial p_j}} + \frac{\partial}{\partial p_j} \underbrace{\dot{p}_j}_{-\frac{\partial H}{\partial q_j}}\Bigg)}_{=0}\Bigg]$$

$$\longrightarrow \operatorname{div}(\varrho \dot{\underline{X}}) \overset{(14.14)}{=} \{\varrho, H\} \quad (14.37)$$

(14.37) in (14.36) $\longrightarrow$
$$\boxed{0 = \frac{\partial \varrho}{\partial t} + \{\varrho, H\}} \quad (14.38)$$

... Liouvillesche Satz

(i) (14.38) $\overset{\nearrow}{=}$ $\frac{d\varrho}{dt} = 0$ ... Dichte entlang Systembahn ist konstant = „inkompressible"
(14.13) Flüssigkeit



(ii) thermodynam. Gleichgewicht:

$$\boxed{\frac{\partial \varrho}{\partial t} = 0 \longrightarrow \{\varrho, H\} = 0} \quad (14.39)$$

mögliche Lsg: $\varrho = \varrho(H) = \varrho\left(H(\{q_j\}, \{p_j\})\right) = \sum_n \varrho_n H^n$

$$\{H^n, H\} = 0$$

Bsp: kanonische Ensemble: $\boxed{\begin{array}{l} \varrho = \frac{1}{Z} e^{-H/k_B T} \\ \text{“}Z = \int d^{6N}X \, e^{-H/k_B T}\text{”} \end{array}}$

14.5 Kanonische Transformationen

[lat: best angepaßt]

• Idee: alle q_j zyklisch & $\frac{\partial H}{\partial t} = 0$

$$\longrightarrow \dot{p}_j = -\frac{\partial H}{\partial q_j} = 0 \longrightarrow p_j = \alpha_j = \text{konst.}$$

$$\longrightarrow H = H(\alpha_1, \dots, \alpha_f) \longrightarrow \dot{q}_j = \frac{\partial H}{\partial \alpha_j} = \omega_j(\alpha_1, \dots, \alpha_f) = \text{konst.}$$

$$\longrightarrow \boxed{q_j = \omega_j t + \beta_j} \quad (14.41)$$

Bem: (i) (14.41) definiert <u>integrable</u> Systeme (... „die Ausnahme")

(ii) die Regel: <u>nicht</u> integrable Systeme (Bsp: 3-Körper-Problem)

Ziel: Trafos $\{q_j, p_j\} \longrightarrow \{Q_k, P_k\}$, so daß alle Q_k zyklisch!

• **Bisher**: $\boxed{\text{Punkttransformation:} \quad Q_k = Q_k(\{q_j\}, t)}$ (14.42)

Beh: $\boxed{\text{Punkttrafos lassen Lagrange Gln. invariant}}$

Bew: Umkehrung: $q_j = q_j(\{Q_k\}, t) \longrightarrow \dot{q}_j = \sum_k \frac{\partial q_j}{\partial Q_k} \dot{Q}_k + \frac{\partial q_j}{\partial t}$ (*)

Zeige: $\frac{d}{dt} \frac{\partial L}{\partial \dot{Q}_k} - \frac{\partial L}{\partial Q_k} = 0$!

$$+\frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial \dot{Q}_k} = 0$$

$$= \sum_j \frac{d}{dt} \Big(\frac{\partial L}{\partial \dot{q}_j} \underbrace{\frac{\partial \dot{q}_j}{\partial \dot{Q}_k}}_{(*)\ \frac{\partial q_j}{\partial Q_k}} \Big) - \sum_j \Big(\frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial Q_k} + \cancel{\frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial Q_k}} \Big)$$

$$\Big(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j}\Big) \frac{\partial q_j}{\partial Q_k} + \cancel{\frac{\partial L}{\partial \dot{q}_j} \underbrace{\frac{d}{dt} \frac{\partial q_j}{\partial Q_k}}_{\frac{\partial \dot{q}_j}{\partial Q_k}}}$$

$$= \sum_j \underbrace{\Big(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} \Big)}_{=0} \frac{\partial q_j}{\partial Q_k} = 0 \checkmark$$

• $\boxed{\text{kanonische Trafos:} \quad Q_k = Q_k(\{q_j\}, \{p_j\}, t)}$

$$P_k = P_k(\{q_j\}, \{p_j\}, t) \qquad (14.43)$$

wenn Hamiltonsche Bewgl. forminvariant bleiben, mit neuer Hamiltonfunktion $\bar{H} = \bar{H}(\{Q_k\}, \{P_k\}, t)$

also: $$L(\{q_j\}, \{\dot{q}_j\}, t) \overset{\text{kanon}}{\underset{\text{Trafo}}{=\!=\!=}} L(\{Q_k\}, \{\dot{Q}_k\}, t) + \frac{d}{dt} F \qquad (14.44)$$

vgl. Kap. 13.5 c)

Legendre $\overset{\longleftrightarrow}{\text{Trafo}}$ $$\boxed{\sum_j p_j \dot{q}_j - H = \sum_j P_j \dot{Q}_j - \bar{H} + \frac{d}{dt} F} \qquad (14.45)$$

• <u>erzeugende Funktion F</u>

(i) bestimmt kanonische Trafo!

(ii) soll also von $\underbrace{\{q_j\}, \{p_j\}, \{Q_j\}, \{P_j\}}$ abhängen

2f unabh.
2f abhängige Variable [vgl. (14.43)]

$\longrightarrow$ <u>4 Möglichkeiten</u>

$$F_1(\{q_j\}, \{Q_j\}, t) \qquad F_2(\{q_j\}, \{P_j\}, t)$$
$$F_3(\{p_j\}, \{Q_j\}, t) \qquad F_4(\{p_j\}, \{P_j\}, t) \qquad (14.46)$$

(i) $F_1(\{q_j\}, \{Q_j\}, t)$:

Bilde $$\frac{d}{dt} F_1 = \sum_j \left(\frac{\partial F_1}{\partial q_j} \dot{q}_j + \frac{\partial F_1}{\partial Q_j} \dot{Q}_j \right) + \frac{\partial F_1}{\partial t}$$

in (14.45) & Koeff. vgl. bei $\dot{q}_j, \dot{Q}_j$, da $\{q_j\}, \{Q_j\}$ unabh. Variable

für $\dot{q}_j$ → $\quad p_j = \frac{\partial F_1}{\partial q_j}$ . . . . . . $\xrightarrow{\text{Umkehrg}}$ $Q_k = Q_k(\{q_j\}, \{p_j\}, t)$!

für $\dot{Q}_j$ → $\quad P_j = -\frac{\partial F_1}{\partial Q_j}$ . . . . . → $P_k = P_k(\{q_j\}, \{Q_j\}, t)$

$= P_k(\{q_j\}, \{p_j\}, t)$!

Rest → $\quad \bar{H} = H + \frac{\partial F_1}{\partial t}$

$$\boxed{\begin{aligned} p_j &= \frac{\partial F_1}{\partial q_j} \\ P_j &= -\frac{\partial F_1}{\partial Q_j} \\ \bar{H} &= H + \frac{\partial F_1}{\partial t} \end{aligned}}$$

... F_1 <u>erzeugt kanonische Trafo</u>

(ii) $F_2(\{q_j\}, \{P_j\}, t) \overset{\text{Leg.}}{\underset{\text{trafo}}{=}} F_1 + \sum_j P_j Q_j$ (14.48)

[$y(x)$, $p = \frac{\partial y}{\partial x}$

$\rightarrow \psi(p) = y(x) - px$

$x = -\frac{\partial \psi}{\partial p}$] →

$$\boxed{\begin{aligned} p_j &= \frac{\partial F_2}{\partial q_j} \\ Q_j &= +\frac{\partial F}{\partial P_j} \\ \bar{H} &= H + \frac{\partial F_2}{\partial t} \end{aligned}} \quad (14.49)$$

... F_2 erzeugt kanon. Trafo

(iii) $F_3(\{p_j\}, \{Q_j\}, t) \overset{\text{Leg.}}{\underset{\text{trafo}}{=}} F_1 - \sum_j p_j q_j$ (14.50) →

$$\boxed{\begin{aligned} q_j &= -\frac{\partial F_3}{\partial p_j} \\ P_j &= -\frac{\partial F_3}{\partial Q_j} \\ \bar{H} &= H + \frac{\partial F_3}{\partial t} \end{aligned}} \quad (14.51)$$

(iv) $F_4(\{p_j\},\{P_j\},t) \underset{\text{trafo}}{\overset{\text{Leg.}}{=}} F_1 + \sum_j P_j Q_j - \sum_j p_j q_j$ … F_3 erzeugt kanon. Trafo

$$\longrightarrow \boxed{\begin{aligned} q_j &= -\frac{\partial F_4}{\partial p_j} \\ Q_j &= \frac{\partial F_4}{\partial P_j} \\ \bar{H} &= H + \frac{\partial F_4}{\partial t} \end{aligned}} \quad (14.53)$$

… F_4 erzeugt kanon Trafo

Beispiele:

(i) identische Trafo: $F_2 = \sum_j q_j P_j \longrightarrow p_j = \frac{\partial F_2}{\partial q_j} = P_j$

$$Q_j = \frac{\partial F_2}{\partial P_j} = q_j \quad !$$

$$\bar{H} = H$$

(ii) $\boxed{\text{Punkttrafo} = \text{kanon. Trafo}}$

$$F_2 = \sum_j f_j(\{q_k\},t)\, P_j \xrightarrow[Q_j = \frac{\partial F_2}{\partial P_j}]{} Q_j = f_j(\{q_k\},t), \quad \bar{H} = H + \frac{\partial F_2}{\partial t}$$

(iii) $F_1 = \sum_j q_j Q_j \longrightarrow$

$$\left.\begin{aligned} p_j &= \frac{\partial F_1}{\partial q_j} = Q_j \\ P_j &= -\frac{\partial F_1}{\partial Q_j} = -q_j \end{aligned}\right\} \text{Vertauschung von Koord. \& Impulse}$$

(iv) harmonischer Oszillator: $(x,p) = (q,p)$

$$\boxed{H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2} \quad (14.54)$$, mit Federkonst: $f = m\omega^2$

$$F_1 = \frac{m}{2}\omega q^2 \cot Q \longrightarrow p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q \quad (1)$$

$$P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2}\frac{1}{\sin^2 Q} \quad (2)$$

$$\left.\begin{aligned} &(2) \longrightarrow \quad q = \sqrt{\frac{2P}{m\omega}} \sin Q \quad (3) \\ &(3) \text{ in } (1) \longrightarrow \quad p = \sqrt{2m\omega P} \cos Q \quad (4) \end{aligned}\right\} \text{kanon. Umkehrtrafo !!}$$

(3), (4) in H: $\quad H = \bar{H} = \omega P \cos^2 Q + \omega P \sin^2 Q$

$$\longrightarrow \boxed{H = \omega P \longleftrightarrow \text{Impuls: } P = \frac{\text{Energie } E = H}{\omega}}$$

Q ... zykl. Koord. $\longrightarrow$ Bewgl. $\dot{Q} = \frac{\partial H}{\partial P} = \omega$

$\downarrow$

$P = \text{const.}$

$$\longrightarrow \boxed{Q = \omega t + \alpha}$$

$$\longrightarrow \boxed{q \overset{(3)}{=} \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \alpha)} \quad \checkmark$$