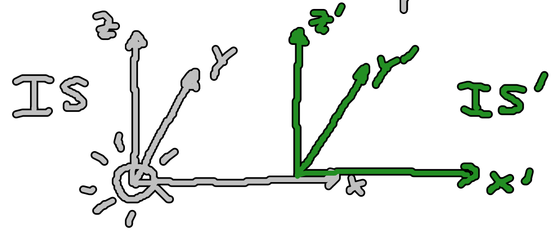
16.3 Der Minkowski-Raum

- Gedanken-Experiment:



$\underbrace{t=0}_{IS=IS'}$

$$\Longrightarrow \quad \boxed{-c^2t^2 + x^2 + y^2 + z^2 = -c^2t'^2 + x'^2 + y'^2 + z'^2} \quad (16.9)$$

... Norm im Minkowski-Raum ist Lorentz-Skalar

- **Minkowski-Raum**: = 4D Raumzeit

(kontravarianter) Vierervektor: $\begin{pmatrix} ct \\ \underline{r} \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$, Komp. x^α, $\alpha = 0,1,2,3$, mit $x^0 = ct$

- Minkowski Norm: $\Delta s^2 = x^\alpha g_{\alpha\beta} x^\beta = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$ (16.11)

mit $g_{\alpha\beta}$ Elemente von $\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$ ... metrischer Tensor (16.12)

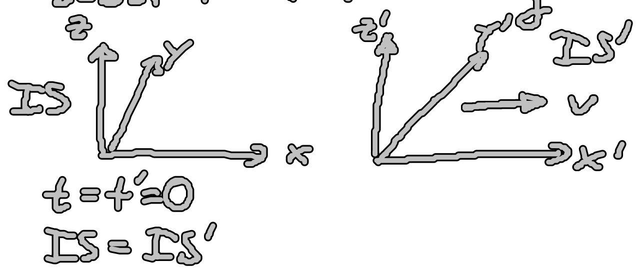
- kovarianter Vierervektor: $\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$ 'Komp: $x_\alpha = g_{\alpha\beta} x^\beta$ (16.13)

- Minkowski-Norm: $\Delta s^2 = x^\alpha x_\alpha$ (16.14)

Δs ... "Abstand des Raum-Zeit-Pkts vom Ursprung"

16.4 Die Lorentz-Trafo

- Trafo von IS → IS'
- **Def:** $\boxed{\text{Die Minkowski-Norm (16.9) ist invariant unter Lorentz-Trafos}}$
- "boost" in x-Richtung: also $y = y'$; $z = z'$



allg. lineare Trafo:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \quad (16.15)$$

⋮

$$\boxed{\begin{aligned} x' &= \gamma(x - \beta ct) \\ ct' &= \gamma(ct - \beta x) \\ \beta &= \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \end{aligned}} \quad (16.16)$$

Lorentz-Trafo für boost in x-Richtung

NB: (1) $\beta = \frac{v}{c} \ll 1 \longrightarrow \beta \approx 0,\ \gamma \approx 1$

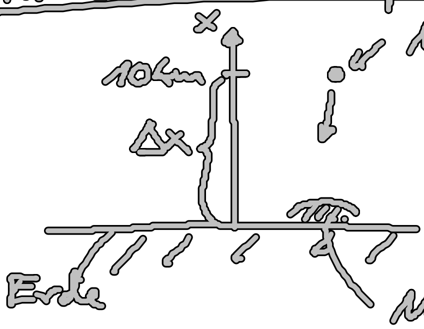
$$\xRightarrow{(16.16)} \left.\begin{aligned} x' &= x - vt \\ t' &= t \end{aligned}\right\} \text{Galilei-Trafo. ✓}$$

(2) synchronisierte Uhren in IS:

$$\longrightarrow ct' \underset{\substack{\uparrow \\ ct=0}}{=} -\gamma\beta x$$

... nicht synchronisiert in IS'! Gleichzeitigkeit ist relativ!

- illustratives Bsp:



Myon: mittlere Lebensdauer $\tau = 2\cdot 10^{-6}\,s$

Messung der Myonen

(1) ohne SRT: $v_\mu \approx c = 3\cdot 10^8 \frac{m}{s}$

$\longrightarrow s = v_\mu \tau = 600\,m \ll 10\,km$ ↯

(2) mit SRT: IS = Erde

IS' = Myon bei $x' = 0$, „innere Uhr": $\tau = \Delta t'$

a) in IS: Flugzeit Δt?

(16.16): $\longrightarrow$ $c\Delta t' = \gamma(c\Delta t - \beta\Delta x)$ (1)

$\Delta x' = 0 = \gamma(\Delta x - \beta c\Delta t)$ (2)

(2) $\longrightarrow$ $\Delta x = \beta c \Delta t$

in (1) $\longrightarrow$ $c\Delta t' = \gamma c \Delta t \underbrace{(1-\beta^2)}_{\gamma^{-2}}$

$$\longrightarrow \boxed{\Delta t' = \sqrt{1-\frac{v^2}{c^2}}\,\Delta t < \Delta t}$$

... Zeitdilatation = bewegte Uhren gehen langsamer

Bsp: $v = \frac{999}{1000}c \longrightarrow \sqrt{1-\frac{v^2}{c^2}} \approx \frac{5}{100}$ $\quad \Delta t = \frac{100}{5}\underset{\Delta t'}{\tau} = 4\cdot 10^{-5}\,s$

$\longrightarrow \Delta x \approx c\Delta t = 12\,km$ ✓

b) in IS': Flugweg $\Delta x'$? Messe Δx in IS' für $\Delta t' = 0$

(16.16) $\quad \Delta x' = \gamma(\Delta x - \beta c \Delta t)$ (3)

$0 = \gamma(c\Delta t - \beta\Delta x)$ (4)

$(4) \longrightarrow c\Delta t = \beta \Delta x$ in (3)

$$\longrightarrow \Delta x' = \gamma \Delta x \underbrace{(1-\beta^2)}_{\gamma^{-2}}$$

$$\longrightarrow \boxed{\Delta x' = \sqrt{1-\frac{v^2}{c^2}}\, \Delta x < \Delta x} \quad (16.18)$$

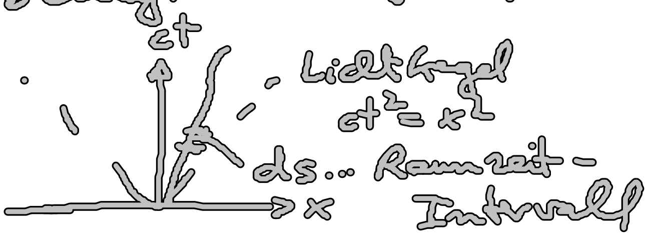
... Lorentzkontraktion = bewegte Maßstäbe sind kürzer

Myon: $\Delta x' = \frac{1}{20}\, 10\,\text{km} = 500\,\text{m}$

17. Relativistische Mechanik

17.1. Eigenzeit

• bewegte Uhr: (16.17) $c\,d\tau = c\,dt' = \sqrt{1-\frac{v^2}{c^2}}\, c\,dt$



$$= [c^2dt^2 - v^2dt^2]^{1/2}$$

$$= [\underbrace{c^2dt^2 - d\underline{r}^2}_{-ds^2}]^{1/2}$$

$-ds^2$ ... Lorentzskalar

$$\longrightarrow \boxed{\tau = \int \sqrt{1-\frac{v^2}{c^2}}\, dt = \int \gamma^{-1}\, dt} \quad (16.19)$$

... Eigenzeit = Lorentzskalar

17.2. Impulssatz

• Regeln: (i) verknüpfe Vierervektoren

(ii) $\frac{v}{c} \ll 1 \longrightarrow$ Newt. Mechanik

• Differenzvektor: dx^α, $\begin{pmatrix} c\,dt \\ d\underline{r} \end{pmatrix}$ (16.20)

• Vierer-Geschw:

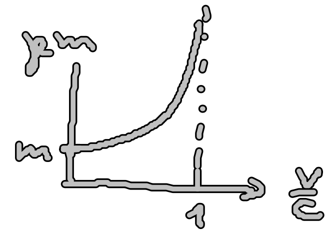
$$u^\alpha = \frac{dx^\alpha}{d\tau} \underset{(16.19)}{=} \gamma \frac{dx^\alpha}{dt}, \quad \begin{pmatrix} \gamma c \\ \gamma \frac{d\underline{r}}{dt} \end{pmatrix} = \begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix} \quad (16.21)$$

• Vierer-Impuls:

$$\boxed{p^\alpha = \underset{\uparrow}{m} u^\alpha \;, \quad \begin{pmatrix} m\gamma c \\ m\gamma \underline{v} \end{pmatrix} = \begin{pmatrix} p^0 \\ \underline{p} \end{pmatrix}} \qquad (16.22)$$

Masse = Skalar

$$\boxed{\underline{p} = \underbrace{\frac{m}{\sqrt{1-\frac{v^2}{c^2}}}}_{\text{geschw. abhängige Impulsmasse!}} \underline{v}}$$



• Impulssatz:

$$\boxed{\frac{dp^\alpha}{d\tau} = \underbrace{K^\alpha}_{\text{Viererkraft!?}}} \qquad (16.24)$$

(i) räumliche Komp. $(\alpha = 1,2,3)$

$$\frac{d\underline{p}}{d\tau} = \gamma \frac{d\underline{p}}{dt} \underset{\text{„Newton"}}{=} \boxed{\gamma \underline{F} = \underline{K}} \qquad (16.25)$$

macht Sinn: $\frac{v}{c} \to 0 \Rightarrow \gamma \approx 1$: $\quad \frac{dm\underline{v}}{dt} = \underline{F}$!

(ii) zeitl. Komp.: $(\alpha = 0)$

– Geschwindigkeitsquadrat:

$$u^\alpha u_\alpha \underset{(16.21)}{=} \gamma^2 \underbrace{\left(-c^2 + \underline{v}^2\right)}_{-c^2\left(1-\frac{v^2}{c^2}\right)} = -c^2 \qquad (16.26)$$

– Betrachte: $\frac{1}{2} m \frac{d}{d\tau} u^\alpha u_\alpha = 0 = m u_\alpha \frac{du^\alpha}{d\tau} = u_\alpha \frac{dp^\alpha}{d\tau}$

$$\longrightarrow \quad 0 = u_\alpha K^\alpha = -\underbrace{u^0}_{\gamma c} K^0 + \gamma \underline{v} \cdot \underbrace{\underline{K}}_{\gamma \underline{F}}$$

also:

$$\boxed{\begin{aligned} K^0 &= \frac{\gamma}{c}\, \underline{v} \cdot \underline{F} \\ &= \frac{\gamma}{c}\, W \end{aligned}} \qquad (16.26)$$

pro Zeiteinheit verrichtete Arbeit

– Energiesatz! aus 0-ter Komp. vom Impulssatz

$$\frac{dp^0}{d\tau} = \gamma \frac{dp^0}{dt} = K^0 = \frac{\gamma}{c}\, \underline{v} \cdot \underline{F}$$

$$\longrightarrow \quad \boxed{\frac{dE}{dt} = \underline{v} \cdot \underline{F} \quad \text{mit } E = cp^0 \underset{(16.22)}{=} m\gamma c^2} \qquad (16.27)$$

... Energie eines Teilchens

<u>Grenzfall:</u> $\frac{v}{c} \ll 1$

$$E = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}\, c^2 \approx mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)$$

$$\longrightarrow \quad \boxed{E = \underbrace{mc^2}_{\text{Ruhe-energie}} + \underbrace{\frac{1}{2} m v^2}_{\text{kinet. Energie}}} \qquad (16.28)$$