3.3 TCP - Invarianz

Invarianzeigenschaften der Maxwell-Gln.:
- Lorentz-Invarianz (s. Kap. 6)
- TCP-Invarianz

außerdem:
- linear in $\underline{E}, \underline{B}, \underline{D}, \underline{H}$ (Superpositionsprinzip)
- 1. Ordnung in t (Kausalitätsprinzip:
  $\underline{E}, \underline{B}, \underline{D}, \underline{H}$ zur Zeit $t=0$ soll den Zustand für $t>0$ vollständig festlegen)

Zeitumkehr $T: t \to t' = -t$
Ladungsumkehr $C: Q \to Q' = -Q$
Paritätsumkehr $P: \underline{r} \to \underline{r}' = -\underline{r}$

NB: Schwache WW verletzt die Paritätserh. (Pauli 1957)
(P, PC, C, T ist verletzt, aber PCT ist erhalten)

(i) Zeitumkehr-Transformation

$$T_g := \{ T\text{-invar. Obs. } A : TA = A \}$$

$$= \{ \underline{r},\ d\underline{r},\ \underline{a} := \frac{d^2\underline{r}}{dt^2},\ m,\ q,\ \rho = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V},$$

$$\underline{F} = m\underline{a},\ \underline{E} = \frac{1}{q}\underline{F},\ \text{Pot. } \phi, \dots \}$$

„gerade" unter T

$$T_u := \{ A : TA = -A \}$$ „ungerade" unter T

$$= \{ \underline{v} = \frac{d\underline{r}}{dt},\ \underline{j} = \rho\underline{v},\ \underline{B},\ \underline{A}, \dots \}$$

$\underline{B} = \nabla \times \underline{A}$

$\hat{\mathcal{T}}_u \quad \hat{\mathcal{T}}_g$

$\Rightarrow$ T-Invarianz der Maxwell-Gln.:

$$T: \{\underline{\nabla}_r \times \underline{E} + \dot{\underline{B}} = 0\} \longrightarrow \{\underline{\nabla} \times \underline{E} + \dot{\underline{B}} = 0\}$$

$$T: \{\underline{\nabla} \cdot \underline{B} = 0\} \longrightarrow \{-\underline{\nabla} \cdot \underline{B} = 0\}$$

$$T: \{\epsilon_0 \underline{\nabla}_r \cdot \underline{E} = \rho\} \longrightarrow \{\epsilon_0 \underline{\nabla}_r \cdot \underline{E} = \rho\}$$

$$T: \{\underline{\nabla} \times \underline{H} - \dot{\underline{D}} = \underline{j}\} \longrightarrow \{-\underline{\nabla} \times \underline{H} + \dot{\underline{D}} = -\underline{j}\}$$

Kontinuitätsgl.:

$$T: \{\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{j} = 0\} \longrightarrow \{-\frac{\partial \rho}{\partial t} - \underline{\nabla} \cdot \underline{j} = 0\}$$

Die Maxwell-Gln. sind forminvariant!

(ii) Ladungsumkehr (-konjugation)

$C_g = \{\underline{F}, m, \underline{r}, \underline{v}, \underline{a}, \ldots\}$ gerade unter C

$C_u = \{\underline{E} = \frac{1}{q}\underline{F}, \underline{B}, \rho, \underline{j}, \ldots\}$ ungerade unter C

da $\underline{F} = q\underline{v} \times \underline{B}$

$\Rightarrow$ C-Invarianz der Maxwell-Gln.:

$$C: \{\underline{\nabla} \times \underline{E} + \dot{\underline{B}} = 0\} \longrightarrow \{-\underline{\nabla} \times \underline{E} - \dot{\underline{B}} = 0\}$$

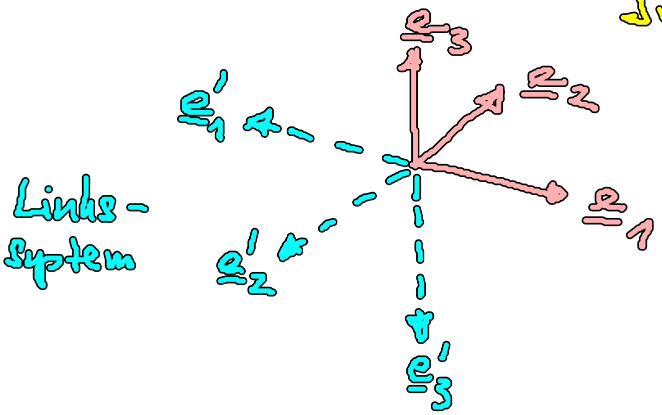
$$C: \{\underline{\nabla} \cdot \underline{B} = 0\} \longrightarrow \{-\underline{\nabla} \cdot \underline{B} = 0\}$$

$$C: \{\epsilon_0 \nabla\cdot\underline{E} = \rho\} \rightarrow \{-\epsilon_0 \nabla\cdot\underline{E} = -\rho\}$$

$$C: \{\nabla\times\underline{H} - \dot{\underline{D}} = \underline{j}\} \rightarrow \{-\nabla\times\underline{H} + \dot{\underline{D}} = -\underline{j}\}$$

$$C: \{\frac{\partial\rho}{\partial t} + \nabla\cdot\underline{j} = 0\} \rightarrow \{-\frac{\partial\rho}{\partial t} - \nabla\cdot\underline{j} = 0\}$$

(iii) <u>Paritätsumkehr</u> (Räumliche Spiegelung am Ursprung, Inversion, Vertauschung rechts $\leftrightarrow$ links)



$P\underline{r} = -\underline{r}$ „polarer" Vektor

aber $P(\underline{a}\times\underline{b}) = (-\underline{a})\times(-\underline{b}) = \underline{a}\times\underline{b}$ „axialer" Vektor

P-invariant (Pseudovektor)

Seien $\underline{a}, \underline{b}$ polar $\rightarrow$ $\underline{a}\times\underline{\omega}$ polar

$\underline{\omega}, \underline{\sigma}$ axiale $\quad \underline{a}\times\underline{b}, \underline{\omega}\times\underline{\sigma}$ axial

$\underline{a}\cdot\underline{b}$ skalar $P(\underline{a}\cdot\underline{b}) = \underline{a}\cdot\underline{b}$

$\underline{a}\cdot\underline{\omega}$ pseudoskalar $P(\underline{a}\cdot\underline{\omega}) = -\underline{a}\,\underline{\omega}$

$P_g = \{$ Skalare m, q; axiale Vektoren $\underline{B}$ $\}$ gerade

$$\underset{P_u}{\underline{F}} = \underset{P_g}{q}\,\underset{P_u}{\underline{v}} \times \underset{P_g}{\underline{B}}$$

$P_u\{$ polare Vektoren $\underline{r}, d\underline{r}, \underline{v}, \underline{a}, \underline{F}, \underline{E} = \frac{1}{q}\underline{F}, \underline{j} = \rho\underline{v}, \underline{A}$,
Pseudoskalar $\underset{u}{\nabla}\cdot\underset{g}{\underline{B}}$ $\}$

$$\underset{g}{\underline{B}} = \underset{u}{\nabla}\times\underset{u}{\underline{A}}$$

<u>P-Invarianz der Maxwell-Gln.:</u>

$$P: \{\nabla\times\underline{E} + \dot{\underline{B}} = 0\} \rightarrow \{\nabla\times\underline{E} + \dot{\underline{B}} = 0\}$$

$$P: \{\nabla\cdot\underline{B} = 0\} \rightarrow \{-\nabla\cdot\underline{B} = 0\} \; !$$

$P: \{\varepsilon_0 \nabla\cdot\underline{E} = \rho\} \rightarrow \{\varepsilon_0 \nabla\cdot\underline{E} = \rho\}$

$P: \{\nabla\times\underline{H} - \dot{\underline{D}} = \underline{j}\} \rightarrow \{-\nabla\times\underline{H} + \dot{\underline{D}} = -\underline{j}\}$

$P: \{\frac{\partial}{\partial t}\rho + \nabla\cdot\underline{j} = 0\} \rightarrow \{\frac{\partial}{\partial t}\rho + \nabla\cdot\underline{j} = 0\}$

<u>Lorentzkraft</u> $\underline{F} = q(\underline{E} + \underline{v}\times\underline{B})$

soll aus einem Extremal-Prinzip (Hamilton'sches Prinzip) ableitbar sein.

$\Rightarrow$ Zus.hang zwischen $\underline{E}, \underline{B}$ und $\phi, \underline{A}$

Suche Lagrange-Fkt. $L(\underline{r}, \underline{v}, t)$, so dass Lagrange-Gln.

$$\frac{d}{dt}\frac{\partial L}{\partial v_k} - \frac{\partial L}{\partial x_k} = 0$$

ergibt: $m\ddot{\underline{r}} = q[\underline{E}(\underline{r},t) + \underline{v}\times\underline{B}(\underline{r},t)]$ (*)

Lagrange-Fkt. $\boxed{L = \frac{m}{2}v^2 + q[\underline{v}\cdot\underline{A}(\underline{r},t) - \phi(\underline{r},t)]}$

Tatsächlich gilt

$$p_k = \frac{\partial L}{\partial v_k} = \underbrace{m v_k}_{\text{kinet. Impuls}} + q A_k(\underline{r},t) = \text{„}\underline{\text{kanon. Impuls}}\text{"}$$

$$\frac{d}{dt}\frac{\partial L}{\partial v_k} = m\ddot{x}_k + q\frac{d}{dt}A_k(\underline{r},t)$$

totale Zeitableit. längs einer Bahn $\underline{r}(t)$

$$= m\ddot{x}_k + q\left(\frac{\partial}{\partial t}A_k + \sum_\ell \frac{\partial A_k}{\partial x_\ell}\dot{x}_\ell\right)$$

$$= m\ddot{x}_k + q\left(\frac{\partial}{\partial t} + \underline{v}\cdot\nabla\right)A_k$$

$$\frac{\partial L}{\partial x_k} = q\left[\frac{\partial}{\partial x_k}(\underline{v}\cdot\underline{A}) - \frac{\partial}{\partial x_k}\phi\right]$$

$$\Rightarrow 0 = \frac{d}{dt}\frac{\partial L}{\partial v_k} - \frac{\partial L}{\partial x_k} = m\ddot{x}_k + q\frac{\partial}{\partial t}A_k + q\underbrace{\left[(\underline{v}\cdot\underline{\nabla})A_k - \frac{\partial}{\partial x_k}(\underline{v}\cdot\underline{A})\right]}_{-[\underline{v}\times(\underline{\nabla}\times\underline{A})]_k} + q\frac{\partial}{\partial x_k}\phi$$

$$0 = m\underline{\ddot{r}} + q\left[\frac{\partial}{\partial t}\underline{A} + \underline{\nabla}\phi - \underline{v}\times(\underline{\nabla}\times\underline{A})\right]$$

Vergleich mit Lorentzkraft (*) :

$$\boxed{\begin{aligned} \underline{E}(\underline{r},t) &= -\frac{\partial}{\partial t}\underline{A}(\underline{r},t) - \underline{\nabla}\phi(\underline{r},t) \\ \underline{B}(\underline{r},t) &= \underline{\nabla}\times\underline{A}(\underline{r},t) \end{aligned}}$$

damit sind die homog. Maxwell-Gln. automatisch erfüllt!

$$\operatorname{div}\underline{B} = \underline{\nabla}\cdot(\underline{\nabla}\times A) \equiv 0$$

$$\underline{\nabla}\times\underline{E} = -\frac{\partial}{\partial t}\underbrace{\underline{\nabla}\times\underline{A}}_{\underline{B}} - \underbrace{\underline{\nabla}\times(\underline{\nabla}\phi)}_{\equiv 0} = -\frac{\partial}{\partial t}B$$

3.4 Energiebilanz

Die Maxwell-Gln. enthalten die Kontin.gl. für ρ :

$$\frac{\partial}{\partial t}\rho + \underline{\nabla}\cdot\underline{j} = 0 \quad \text{(Ladungserhaltung)}$$

Frage: weitere Erhaltungssätze für extensive phys. Obs., z.B. Energie, Impuls, Drehimpuls?

(Ext.: additiv bei Systemzusetzung
Intensiv: z.B. Temp.)

Energietransport durch das el.magn. Feld:

$$\underline{\nabla}\times\underline{E} + \frac{\partial}{\partial t}\underline{B} = 0 \quad |\cdot\underline{H}$$

$$\underline{\nabla}\times\underline{H} - \frac{\partial}{\partial t}\underline{D} = \underline{j} \quad |\cdot\underline{E}$$

$$\Rightarrow \underbrace{\underline{H}\cdot(\underline{\nabla}\times\underline{E}) - \underline{E}(\underline{\nabla}\times\underline{H})}_{\underline{\nabla}\cdot(\underline{E}\times\underline{H})} + \underbrace{\underline{H}\frac{\partial}{\partial t}\underline{B}}_{\frac{1}{\mu_0}\underbrace{\underline{B}\frac{\partial}{\partial t}\underline{B}}_{\frac{1}{2}\frac{\partial}{\partial t}\underline{B}^2}} + \underbrace{\underline{E}\frac{\partial}{\partial t}\underline{D}}_{\varepsilon_0\underbrace{\underline{E}\frac{\partial}{\partial t}\underline{E}}_{\frac{1}{2}\frac{\partial}{\partial t}\underline{E}^2}} = -\underline{j}\cdot\underline{E}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}w + \underline{\nabla}\cdot\underline{S} = -\underline{j}\cdot\underline{E}}$$

Kontin.gl.
= Bilanzgl. für Energietransport

$$w := \frac{\varepsilon_0}{2}\underline{E}^2 + \frac{1}{2\mu_0}\underline{B}^2 = \frac{1}{2}(\underline{E}\cdot\underline{D} + \underline{B}\cdot\underline{H})$$ Energiedichte

$$\underline{S} := \underline{E}\times\underline{H}$$ Energiestromdichte des el.magn. Feldes (Poynting-Vektor)

$$\sigma = -\underline{j}\cdot\underline{E}$$ Quelldichte der Feldenergie (Leistungsdichte)

$\underline{j}\cdot\underline{E} > 0$ Abnahme der Feldenergie

$\underline{j}\cdot\underline{E} < 0$ Zunahme der Feldenergie