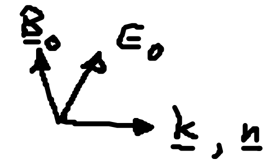
Energiedichte der el. magn. Welle:

Sei $\underline{E}_0$ reell $\Longrightarrow$ $\underline{E}(\underline{r},t) = \underline{E}_0 \cos(\underline{k}\cdot\underline{r} - \omega t)$

$$\underline{B}(\underline{r},t) = \underbrace{\underline{B}_0}_{\boxed{= \frac{1}{c}\underline{n}\times\underline{E}_0}} \cos(\underline{k}\,\underline{r} - \omega t)$$



Energiedichte $w = \frac{\epsilon_0}{2}\underline{E}^2 + \frac{1}{2\mu_0}\underline{B}^2$

$$= \frac{\epsilon_0}{2}\underline{E}^2 + \frac{1}{2\mu_0}\frac{1}{c^2}\underline{E}^2 \qquad \frac{1}{c^2} = \epsilon_0\mu_0$$

$$= 2\cdot\frac{\epsilon_0}{2}\underline{E}^2$$

Energiestromdichte: $\underline{S} = \frac{1}{\mu_0}\underline{E}\times\underline{B}$

$$= \frac{1}{c\mu_0}\underline{E}\times(\underline{n}\times\underline{E})$$

$$= \sqrt{\frac{\epsilon_0}{\mu_0}}\,E^2\underline{n}$$

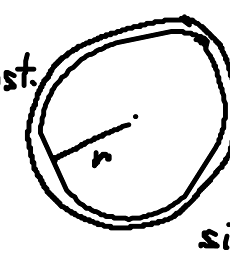
$$= c\,\epsilon_0 E^2\underline{n}$$

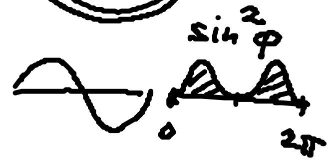
$$= c\,w\,\underline{n} \quad , \underline{n} \equiv \frac{\underline{k}}{|\underline{k}|}$$

Kugelwelle: $\underline{E}(r,t) = \frac{1}{r}\underline{E}_0\, e^{i(kr-\omega t)}$

Energie in Kugelschale mit Radius r u. Dicke dr:

$$W(r) = 4\pi r^2 dr\; \epsilon_0 \underline{E}^2 \underset{\substack{\uparrow\\ \text{Raum/Zeitmittel}\\ \text{über } \exp\{\ \}}}{=} 4\pi r^2 dr\; \epsilon_0 \frac{E_0^2}{r^2}\cdot\frac{1}{2} = \text{const.}$$





4.2 Retardierte Potenziale

Aufgabe: Lösung der inhomog. Wellengln.

$$\boxed{\begin{aligned} \Box\phi &= -\frac{1}{\epsilon_0}\rho \\ \Box\underline{A} &= -\mu_0\underline{j} \end{aligned}}$$

(Lorenz-Eichung)

zu vorgeg. erzeugenden Quellen $\rho(\underline{r},t)$, $\underline{j}(\underline{r},t)$
und Randbed. ϕ, $\underline{A} \to 0$ für $r\to\infty$.

Methode: Green'sche Fkt. $G(\underline{r}-\underline{r}', t-t')$

$$\boxed{\Box u(\underline{r},t) = -f(\underline{r},t)}$$

Fourier-Trafo $\Downarrow$ $\quad u := \begin{cases}\phi \\ \underline{A}\end{cases}$

$\hat{\Box}^{-1} = -\hat{G}$ $\quad f := \begin{cases}\rho/\epsilon_0 \\ \mu_0\underline{j}\end{cases}$

$$\boxed{\hat{u}(\underline{k},\omega) = \hat{G}\cdot\hat{f}(\underline{k},\omega)}$$

Rück-Trafo $\Downarrow$

$$\boxed{\begin{aligned} u(\underline{r},t) &= \int_{\mathbb{R}^3} d^3r' \int_{-\infty}^{\infty} dt'\, G(\underline{r}-\underline{r}', t-t') f(\underline{r}',t') \\ \text{mit } \Box G(\underline{r}-\underline{r}', t-t') &= -\delta(\underline{r}-\underline{r}')\delta(t-t') \end{aligned}}$$

Vgl. Elektrostatik

$$\Delta\phi(\underline{r}) = -\frac{1}{\epsilon_0}\rho(\underline{r})$$

$$\Downarrow$$

$$\hat{\phi}(\underline{k}) = \hat{G}\hat{\rho}, \quad \hat{G} = \frac{1}{\epsilon_0 k^2}$$

$$\Downarrow$$

$$\phi(\underline{r}) = \int d^3r'\, G(\underline{r}-\underline{r}')\rho(\underline{r}')$$

mit $G(\underline{r}-\underline{r}') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\underline{r}-\underline{r}'|}$

$$\Delta G(\underline{r}-\underline{r}') = -\frac{1}{\epsilon_0}\delta(\underline{r}-\underline{r}')$$

Kausalitätsbed.: $G(\underline{r}-\underline{r}', t-t') \overset{!}{=} 0$ für $t'>t$
damit $u(\underline{r},t)$ nur von $f(\underline{r}',t')$ mit $t'<t$
beeinflusst wird.

Fourier-Transformation:

$$f(\underline{r},t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} d^3q \int_{-\infty}^{\infty} d\omega\, \hat{f}(\underline{q},\omega)\, e^{i(\underline{q}\cdot\underline{r}-\omega t)}$$

$$\hat{f}(\underline{q},\omega) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} d^3r \int_{-\infty}^{\infty} dt\, f(\underline{r},t)\, e^{-i(\underline{q}\cdot\underline{r}-\omega t)}$$

Ebenso:

$$u(\underline{r},t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} d^3q \int_{-\infty}^{\infty} d\omega\, \hat{u}(\underline{q},\omega)\, e^{\cdots\cdots\cdots}$$

$$\Box u(\underline{r},t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} d^3q \int_{-\infty}^{\infty} d\omega\, \hat{u}(\underline{q},\omega)\, \underbrace{\Box e^{i(\underline{q}\underline{r}-\omega t)}}_{-(q^2-\frac{\omega^2}{c^2})e^{i(\underline{q}\underline{r}-\omega t)}}$$

$$= -\frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} d^3q \int_{-\infty}^{\infty} d\omega\, \hat{f}(\underline{q},\omega)\, e^{i(\underline{q}\cdot\underline{r}-\omega t)}$$

$$\Longrightarrow \left(q^2-\frac{\omega^2}{c^2}\right)\hat{u}(\underline{q},\omega) = \hat{f}(\underline{q},\omega)$$

$$\Longrightarrow \quad \boxed{\hat{u}(\underline{q},\omega) = \frac{\hat{f}(\underline{q},\omega)}{q^2-\frac{\omega^2}{c^2}}} \qquad \text{d.h. } \hat{G} = \frac{1}{q^2-\frac{\omega^2}{c^2}}$$

Rück-Trafo:

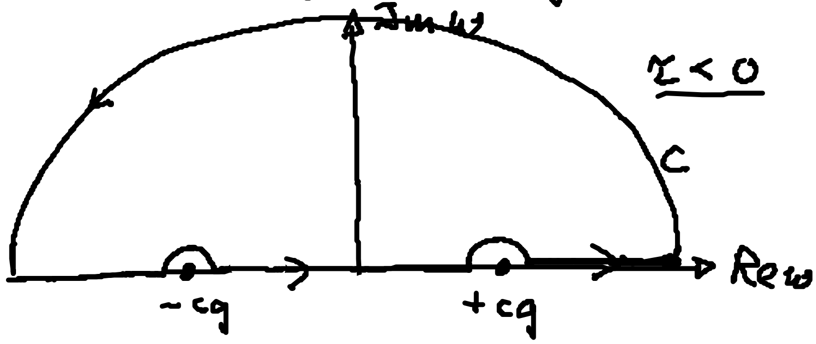
$$u(\underline{r},t) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} d^3q \int_{-\infty}^{\infty} d\omega\, \frac{e^{i(\underline{q}\underline{r}-\omega t)}}{q^2-\frac{\omega^2}{c^2}} \int_{\mathbb{R}^3} d^3r' \int_{-\infty}^{\infty} dt'\, f(\underline{r}',t')\, e^{-i(\underline{q}\underline{r}'-\omega t')}$$

$$= \int_{\mathbb{R}^3} d^3r' \int_{-\infty}^{\infty} dt' \underbrace{\left\{\frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} d^3q \int_{-\infty}^{\infty} d\omega\, \frac{e^{i\underline{q}(\underline{r}-\underline{r}')-i\omega(t-t')}}{q^2-\frac{\omega^2}{c^2}}\right\}}_{G(\underline{r}-\underline{r}',\,t-t')} f(\underline{r}',t')$$

Berechnung der Green'schen Fkt. durch komplexe Integration

Integrand hat Pole bei $\omega = \pm cq$

Green'sche Fkt. wird eindeutig durch Festlegung des Integrationswegs um die Pole herum:



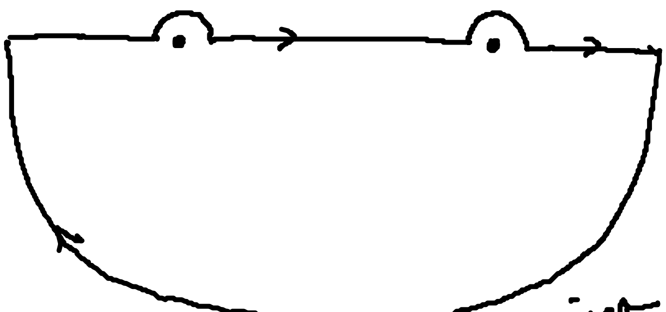
$\underline{\tau < 0}$

Integral über Halbkreis

$\omega = R e^{i\varphi}$ $\qquad 0 \le \varphi \le \pi$

$d\omega = R e^{i\varphi} i\, d\varphi$ $\qquad \boxed{\tau \equiv t - t'}$

$$|e^{-i\omega\tau}| = e^{\underbrace{R(\sin\varphi)}_{>0}\underbrace{\tau}_{<0}} \xrightarrow{R\to\infty} 0$$



$\underline{\tau > 0}$: $\pi \le \varphi \le 2\pi$

$$|e^{-i\omega\tau}| = e^{\underbrace{R(\sin\varphi)}_{<0}\underbrace{\tau}_{>0}} \xrightarrow{R\to\infty} 0$$

$$\Gamma(\underline{q},\tau) := \int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega\tau}}{q^2 - \frac{\omega^2}{c^2}} = \oint_C d\omega \, \frac{e^{-i\omega\tau}}{q^2 - \frac{\omega^2}{c^2}} = 2\pi i \sum_{\text{Pole}} \text{Res}$$

$\underline{\tau < 0}$: Keine Pole im Inneren von C

$$\Rightarrow \Gamma(\underline{q},\tau) = 0 \quad \Rightarrow \quad \boxed{G(\underbrace{\underline{r}-\underline{r}'}_{\underline{s}}, \underbrace{t-t'}_{\tau}) = 0 \text{ für } t < t'}$$

Dies ist die Kausalitätsbedingung!

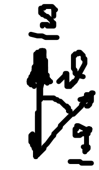
$\underline{\tau > 0}$: $$\Gamma(\underline{q},\tau) = -2\pi i \sum_{\omega = \pm cq} \text{Res} \frac{e^{-i\omega\tau}}{(-\frac{1}{c^2})(\omega - cq)(\omega + cq)}$$

da $\oint dz\, f(z) = 2\pi i \sum \text{Res} f(z) \leftrightarrow$ hier $\circlearrowright$
math. pos. Sinn

$$\Gamma(\underline{q},\tau) = 2\pi i c^2 \left(\frac{e^{-icq\tau}}{2cq} + \frac{e^{icq\tau}}{-2cq} \right)$$

$$G(\underline{s},\tau) = \frac{c}{(2\pi)^3} \int d^3q \, e^{i\underline{q}\cdot\underline{s}} \left(\frac{e^{-icq\tau} - e^{icq\tau}}{-2iq} \right)$$

Auswertung in Kugelkoord.: $d^3q = q^2\,dq\,\sin\vartheta\,d\vartheta\,d\varphi$



$\underline{q}\cdot\underline{s} = qs\cos\vartheta$

$$G(\underline{s},\tau) = \frac{c}{(2\pi)^3}\int_0^\infty dq\,q\,\frac{e^{-icq\tau}-e^{+icq\tau}}{-2i}\underbrace{\int_{-1}^{1} d(\cos\vartheta)\,e^{iqs\cos\vartheta}}_{\frac{e^{iqs}-e^{-iqs}}{iqs}}\underbrace{\int_0^{2\pi} d\varphi}_{2\pi}$$

$\xi := cq$

$$= \frac{1}{2(2\pi)^2 s}\int_0^\infty d\xi\left\{\underbrace{e^{i(\tau-\frac{s}{c})\xi}+e^{-i(\tau-\frac{s}{c})\xi}}\;\underbrace{-e^{i(\tau+\frac{s}{c})\xi}-e^{-i(\tau+\frac{s}{c})\xi}}\right\}$$

Fourier: $\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,e^{ikx} = \delta(x)$

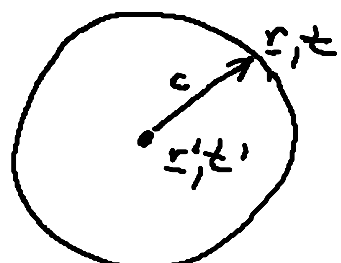
$$= \frac{1}{4\pi s}\left\{\delta\left(\tau-\frac{s}{c}\right) - \underbrace{\delta\left(\tau+\frac{s}{c}\right)}_{0 \text{ für } \tau>0}\right\}$$

Erg.:

$$\boxed{G(\underline{r}-\underline{r}', t-t') = \begin{cases} \frac{1}{4\pi|\underline{r}-\underline{r}'|}\,\delta\left(t-\underbrace{t'-\frac{|\underline{r}-\underline{r}'|}{c}}\right) & t>t' \\ 0 & t<t' \end{cases}}$$

retardierte Green'sche Fkt.

(kausal)



Wellenfront