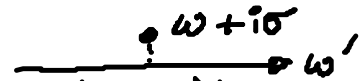
Kramers-Kronig-Relationen

Kausalitätsprinzip $\quad \chi(t) = \Theta(t)\,\chi(t)$

$$\Rightarrow \quad \hat{\chi}(\omega) = \lim_{\sigma\to 0^+} \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega' \, \frac{1}{\omega'-\omega-i\sigma}\,\hat{\chi}(\omega')$$

Integrand hat Pol bei $\omega' = \omega + i\sigma$



Integrationsweg

äquivalenter Int.weg

Zerlegung:

$$\int_{-\infty}^{\infty} d\omega' \, \frac{\hat{\chi}(\omega')}{\omega'-\omega} = \underbrace{\lim_{\varepsilon\to 0^+}\left[\int_{-\infty}^{\omega-\varepsilon} + \int_{\omega+\varepsilon}^{\infty}\right] d\omega' \, \frac{\hat{\chi}(\omega')}{\omega'-\omega}}_{P\int_{-\infty}^{\infty} d\omega' \frac{\hat{\chi}}{\omega'-\omega} \text{ „Hauptwert" (Principal value)}} + \underbrace{\int_{\smile} d\omega' \, \frac{\hat{\chi}(\omega')}{\omega'-\omega}}_{\text{Integral längs Halbkreis mit Radius } \varepsilon \text{ um Pol}}$$

$$\int_{\smile} d\zeta \, \frac{f(\zeta)}{\zeta} = f(0)\int_{\smile} \frac{d\zeta}{\zeta} \underset{\zeta = \varepsilon e^{i\varphi}}{=} f(0)\, i \int_{\pi}^{2\pi} d\varphi = i\pi f(0)$$

halbes Residuum

$$d\zeta = i\zeta\, d\varphi$$

$$\Rightarrow \hat\chi(\omega) = \frac{1}{2\pi i} \mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\hat\chi(\omega')}{\omega' - \omega} + \frac{1}{2}\hat\chi(\omega)$$

$$\Rightarrow \boxed{\hat\chi(\omega) = \frac{1}{\pi i} \mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\hat\chi(\omega')}{\omega' - \omega}}$$

Zerlegung in Re und Im mit $\mathrm{Re}\,\hat\chi(\omega) = \varepsilon'(\omega) - 1$

$\mathrm{Im}\,\hat\chi(\omega) = \varepsilon''(\omega)$

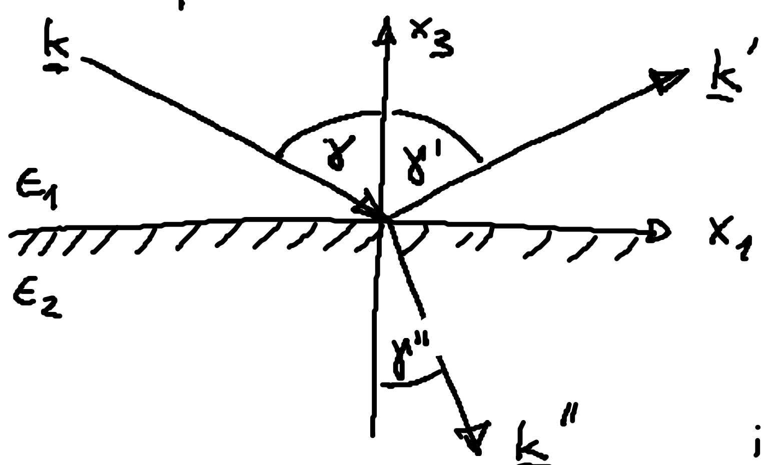
$$\boxed{\begin{aligned} \varepsilon'(\omega) - 1 &= \frac{1}{\pi} \mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\varepsilon''(\omega')}{\omega' - \omega} \\ \varepsilon''(\omega) &= -\frac{1}{\pi} \mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\varepsilon'(\omega') - 1}{\omega' - \omega} \end{aligned}}$$

Kramers-Kronig-Relationen

5.7 Brechung und Reflexion

Wellenausbreitung in geschichteten Medien:

(transparent $\Rightarrow \epsilon_i \in \mathbb{R}$, $i = 1,2$)



$$\boxed{\begin{gathered} \frac{\omega}{c_1} = |\underline{k}| = |\underline{k}'| = \frac{\omega'}{c_1} \\ |\underline{k}''| = \frac{\omega''}{c_2} \\ c_i = \frac{c}{n_i} = \frac{c}{\sqrt{\epsilon_i}} \end{gathered}} \quad i = 1,2$$

Einfallende Welle $\underline{E} = \underline{E}_0 e^{i(\underline{k}\cdot\underline{r} - \omega t)}$

Reflektierte Welle $\underline{E}' = \underline{E}_0' e^{i(\underline{k}'\cdot\underline{r} - \omega' t)}$

Transmittierte Welle $\underline{E}'' = \underline{E}_0'' e^{i(\underline{k}''\cdot r - \omega'' t)}$

Grenzbedingungen für Felder

$\underline{n} \times (\underline{E}^{(1)} - \underline{E}^{(2)}) = 0$ Tang. komp. v. $\underline{E}$ stetig

$\underline{n} \cdot (\underline{D}^{(1)} - \underline{D}^{(2)}) = \cancel{\sigma}$ Norm. komp. v. $\underline{D} = \epsilon_0 \epsilon \underline{E}$

$\underline{n} \times (\underline{H}^{(1)} - \underline{H}^{(2)}) = \cancel{\underline{g}}$ Tangenzialkomp. v. $\underline{H}$

$\underline{n} \cdot (\underline{B}^{(1)} - \underline{B}^{(2)}) = 0$ Normalkomp. v. $\underline{B} = \mu_0 \mu \underline{H}$ stetig

$\mu = 1$

grenzbed. für $\underline{E}$ (linear polarisiert)

$$E_1 + E_1' \Big|_{x_3=0} = E_1'' \Big|_{x_3=0}$$ Tang. komp. stetig

$$\underline{r} = 0 : E_{01} e^{-i\omega t} + E_{01}' e^{-i\omega' t} = E_{01}'' e^{-i\omega'' t} \Rightarrow \boxed{\begin{array}{c} \omega = \omega' = \omega'' \\ E_{01} + E_{01}' = E_{01}'' \end{array}}$$

$$t = 0 : E_{01} e^{i k_1 x_1} + E_{01}' e^{i k_1' x_1} = E_{01}'' e^{i k_1'' x_1} \Rightarrow \boxed{k_1 = k_1' = k_1''}$$

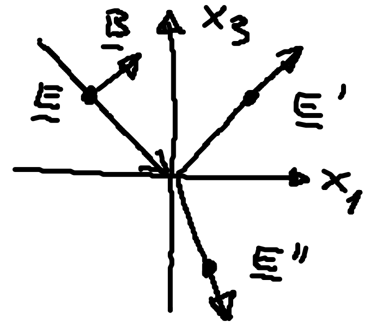
$$\Rightarrow \underbrace{|\underline{k}|}_{\omega/c_1} \sin\gamma = \underbrace{|\underline{k}'|}_{\omega/c_1} \sin\gamma' = \underbrace{|\underline{k}''|}_{\omega/c_2} \sin\gamma''$$

$$\Rightarrow \boxed{\begin{array}{l} \sin\gamma = \sin\gamma' \\ \dfrac{\sin\gamma''}{\sin\gamma} = \dfrac{c_2}{c_1} = \dfrac{n_1}{n_2} \end{array}}$$

Reflexionsgesetz

Brechungsgesetz (Snellius)

<u>Bestimmung der Amplituden:</u>



<u>(a) Polarisation von $\underline{E}$ $\perp$ Einfallsebene</u>

$$E_{01} = E'_{01} = E''_{01} = 0$$
$$E_{03} = E'_{03} = E''_{03} = 0$$

$$(1) \quad \boxed{E_{02} + E'_{02} = E''_{02}} \quad \text{Tang. komp.}$$

Mit $\underline{B}_0 = \frac{c}{\omega}(\underline{k} \times \underline{E}_0) = \frac{c}{\omega} E_{02} \begin{pmatrix} -k_3 \\ 0 \\ k_1 \end{pmatrix}$ folgt für <u>Tang. komp. v. $\underline{B}$</u>:

$$B_{01} + B'_{01} = B''_{01} \Rightarrow k_3 E_{02} + k'_3 E'_{02} = k''_3 E''_{02}$$

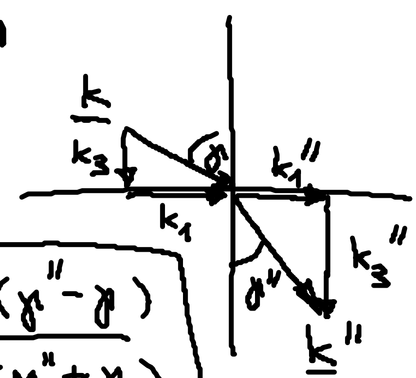
$$\text{Reflexionsgesetz} \Rightarrow k_3 = -k'_3 \Rightarrow \boxed{k_3 (E_{02} - E'_{02}) = k''_3 E''_{02}} \quad (2)$$

$$(1) \text{ in } (2) \Rightarrow k_3 (E_{02} - E'_{02}) \overset{(1)}{=} k''_3 (E_{02} + E'_{02})$$

$$\Rightarrow \boxed{\frac{E'_{02}}{E_{02}} = \frac{k_3 - k''_3}{k_3 + k''_3}, \quad \frac{E''_{02}}{E_{02}} = \frac{2k_3}{k_3 + k''_3}}$$

Drücke k''_3 durch Brechungswinkel γ'' aus!

$$\Rightarrow \quad k''_3 = |\underline{k}''| \cos\gamma'' = |\underline{k}| \underbrace{\left(\frac{n_2}{n_1}\right)}_{\frac{\sin\gamma}{\sin\gamma''}} \cos\gamma''$$
$$k_3 = |\underline{k}| \cos\gamma$$



$$\Rightarrow \boxed{\begin{aligned} \frac{E'_{02}}{E_{02}} &= \frac{\cos\gamma \sin\gamma'' - \sin\gamma \cos\gamma''}{\cos\gamma \sin\gamma'' + \sin\gamma \cos\gamma''} = \frac{\sin(\gamma'' - \gamma)}{\sin(\gamma'' + \gamma)} \\ \frac{E''_{02}}{E_{02}} &= \frac{2 \sin\gamma'' \cos\gamma}{\sin(\gamma'' + \gamma)} \end{aligned}}$$

<u>Fresnel'sche Formeln</u>

Intensitätsverhältnisse:

Zeitmittel des Poyntingvektors $\langle \underline{S} \rangle = \frac{1}{T}\int_0^T dt\,(\underline{E}\times\underline{H}) \sim |E_0|^2$

Reflexionskoeff.:

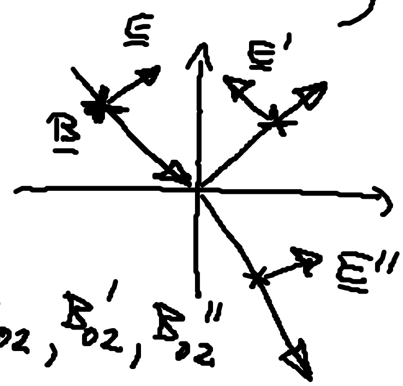
$$R_\perp = \left|\frac{E'_{02}}{E_{02}}\right|^2 = \frac{\sin^2(\gamma''-\gamma)}{\sin^2(\gamma''+\gamma)}$$

Transmissionskoeff.:

$$T_\perp = \left|\frac{E''_{02}}{E_{02}}\right|^2 = \frac{4\sin^2\gamma''\cos^2\gamma}{\sin^2(\gamma''+\gamma)} = 1 - R_\perp$$

$\perp$ polarisiert

(b) Polarisation von $\underline{E}$ $\parallel$ Einfallsebene



$\underline{B}$ $\perp$ Einfallsebene

$\Rightarrow$ analoge Argumentation für $B_{02}, B'_{02}, B''_{02}$ wie in (a)

$$\Rightarrow \boxed{\begin{aligned} &\frac{E'_\parallel}{E_\parallel} = \frac{\tan(\gamma-\gamma'')}{\tan(\gamma+\gamma'')}, \quad \frac{E''_\parallel}{E_\parallel} = \frac{2\sin\gamma''\cos\gamma}{\sin(\gamma+\gamma'')\cos(\gamma''-\gamma)} \\ &R_\parallel = 1 - T_\parallel = \frac{\tan^2(\gamma''-\gamma)}{\tan^2(\gamma''+\gamma)} \end{aligned}}$$

Bem.: (i) Bei Reflexion u. Brechung wird i.a. die Polarisationsrichtung gedreht!

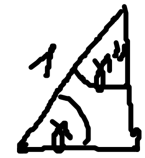
Speziell für:

$\gamma'' + \gamma = \frac{\pi}{2} \Rightarrow \tan(\gamma'' + \gamma) \to \infty \Rightarrow \boxed{R_{\parallel} = 0}$

$\Rightarrow$ reflektierte Welle vollständig polaris. $\perp$ Einfallsebene

γ = Brewster-Winkel γ_B mit $\tan\gamma_B = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$ (*)

$$\tan\gamma = \frac{\sin\gamma}{\cos\gamma} = \frac{\sin\gamma}{\sin(\frac{\pi}{2} - \gamma)} = \frac{\sin\gamma}{\sin\gamma''}$$



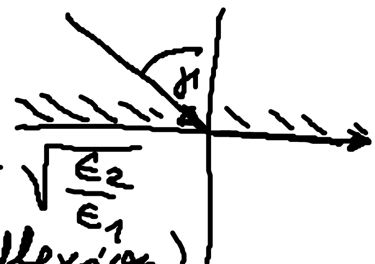
$\sin\gamma$

$\cos\gamma = \sin(\frac{\pi}{2} - \gamma)$

$$= \frac{n_2}{n_1} = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

(ii) Totalreflexion

Sei $\epsilon_2 < \epsilon_1 \Rightarrow$

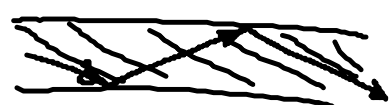


für $\gamma = \gamma_G$ mit $\sin\gamma_G = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$

(Grenzwinkel der Totalreflexion)

$\gamma'' = \frac{\pi}{2}$, $R_{\perp} = R_{\parallel} = 1$

$T_{\perp} = T_{\parallel} = 0$



Opt. Faser

für $\gamma > \gamma_G \Rightarrow k_3'' = \frac{i}{d}$ imaginär

evaneszente Welle $\underline{E}'' = \underline{E}_0'' \underbrace{e^{-|x_3|/d}}_{\text{evaneszent}} e^{i(k_1'' x_1 - \omega t)}$

$R_{\perp} = R_{\parallel} = 1$

$T_{\perp} = T_{\parallel} = 0$