

Organisatorisches:

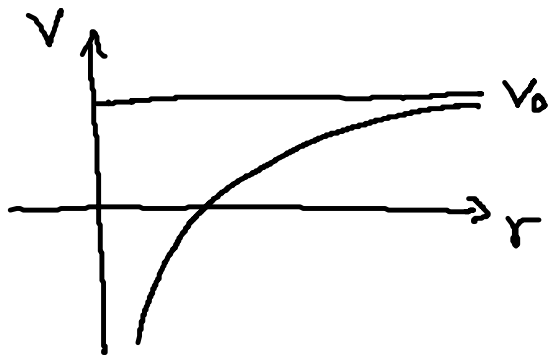
Mo., 15.02.2010 Nachklausur

EW 201 ; 10 Uhr

Klausur:

Aufgabe 1 $V(r) = V_0 - \frac{\alpha}{r^3}$; $\alpha, V_0 > 0$

a) Skizze: $r := |\underline{x}| > 0$



b) Kraft:

$$\underline{F} = -\nabla_{\underline{r}} V$$

$$(i) \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} ; \nabla_{\underline{r}} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$(ii) |\underline{r}| = (x^2 + y^2 + z^2)^{1/2}$$

$$\Rightarrow \underline{F} = - \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \left[- \frac{\alpha}{|\underline{r}|^3} \right]$$

$$= - \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \left(- \frac{\alpha}{(x^2 + y^2 + z^2)^{3/2}} \right)$$

$$= - \frac{3}{2} \alpha \cdot \frac{1}{(x^2 + y^2 + z^2)^{5/2}} \cdot \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$= - \frac{3\alpha}{(x^2 + y^2 + z^2)^{5/2}} \underline{r}$$

$$= -\frac{3a}{|r|^4} \cdot \frac{r}{|r|} = -\frac{3a}{|r|^4} \underline{e}_r \quad 1P.$$

c) Ebene Polarkoordinat:

$$\begin{aligned} x &= r \cos \phi & \Rightarrow & \dot{x} = \dot{r} \cos \phi - r \dot{\phi} \sin \phi \\ y &= r \sin \phi & \Rightarrow & \dot{y} = \dot{r} \sin \phi + r \dot{\phi} \cos \phi \end{aligned} \quad 1P.$$

$$z = z_0 \quad ; \quad \dot{z} = 0$$

$$\Rightarrow T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \dots = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\phi}^2)$$

$$\Rightarrow L = T - V = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{a}{r^3} - V_0 \quad 2P.$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r} \quad ; \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi}$$

$$\Rightarrow \dot{r} = \frac{p_r}{m} \quad \dot{\phi} = \frac{p_\phi}{m r^2} \quad 1P.$$

$$H = \sum_{k=1}^2 \dot{q}_k p_k - L$$

$$= \dot{r} p_r + \dot{\phi} p_\phi - \left\{ \frac{m}{2} \dot{r}^2 + \frac{m}{2} \dot{\phi}^2 + \dots \right\}$$

$$= \frac{p_r^2}{2m} + \frac{p_\phi^2}{2m r^2} - \frac{a}{r^3} + V_0 \quad 1P.$$

$$d) \quad \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \quad 1P.$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{m r^3} - \frac{3a}{r^4}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m r^2} \quad 1P.$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

$$\Rightarrow \phi \text{ ist zyklisch} \quad 1P.$$

Aufgabe 2: $V = -\frac{\alpha}{r}$

a) $T = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$\Rightarrow T = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}$$

$$V = \frac{\alpha}{(x^2 + y^2 + z^2)^{1/2}}$$

$$\Rightarrow H = T + V = \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) - \frac{\alpha}{(x^2 + y^2 + z^2)^{1/2}}$$

1P

b) $\frac{d\vec{A}}{dt} = \{H, \vec{A}\} + \frac{\partial \vec{A}}{\partial t} \stackrel{!}{=} 0$

$$\vec{A} = \frac{1}{m} \vec{p} \times \vec{L} + V(r) \vec{r}$$

$$\frac{\partial \vec{A}}{\partial t} = 0 \Rightarrow \{H, \vec{A}\} = 0$$

$$\Leftrightarrow \frac{d\vec{A}}{dt} = 0$$

1P.

Allgem. Eigenschaft von Zentralpot.:

Bew. in der Ebene $\rightarrow z=0$ und $p_z=0$

$$\vec{A} = \frac{1}{m} \begin{pmatrix} p_x \\ p_y \\ 0 \end{pmatrix} \times \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \times \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \right\} + V(r) \vec{r}$$

$$= \frac{1}{m} \begin{pmatrix} x p_y^2 - y p_x p_y \\ y p_x^2 - x p_x p_y \\ 0 \end{pmatrix} - \frac{\alpha}{(x^2 + y^2)^{1/2}} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$$A_z = 0!$$

2P.

$$H = \frac{1}{2m} (p_x^2 + p_y^2) - \frac{\alpha}{(x^2 + y^2)^{1/2}}$$

$$A_x = \frac{1}{m} (x p_y^2 - y p_x p_y) - \frac{\alpha}{(x^2 + y^2)^{1/2}} x$$

$$\begin{aligned} \{H, A_x\} &= \sum_{k=1}^3 \left(\frac{\partial H}{\partial q_k} \frac{\partial A_x}{\partial p_k} - \frac{\partial H}{\partial p_k} \frac{\partial A_x}{\partial q_k} \right) \\ &= \frac{\partial H}{\partial x} \frac{\partial A_x}{\partial p_x} - \frac{\partial H}{\partial p_x} \frac{\partial A_x}{\partial x} + \frac{\partial H}{\partial y} \frac{\partial A_x}{\partial p_y} - \frac{\partial H}{\partial p_y} \frac{\partial A_x}{\partial y} \end{aligned}$$

$$\frac{\partial H}{\partial x} = \frac{\alpha x}{(x^2 + y^2)^{3/2}} \quad ; \quad \frac{\partial H}{\partial y} = \frac{\alpha y}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial H}{\partial p_x} = \frac{p_x}{m} \quad ; \quad \frac{\partial H}{\partial p_y} = \frac{p_y}{m}$$

$$\frac{\partial A_x}{\partial x} = \frac{p_y^2}{m} - \frac{\alpha}{(x^2 + y^2)^{1/2}} + \frac{\alpha x^2}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial A_x}{\partial y} = -\frac{p_x p_y}{m} + \frac{\alpha x y}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial A_x}{\partial p_x} = -\frac{y p_y}{m} \quad ; \quad \frac{\partial A_x}{\partial p_y} = \frac{2x p_y}{m} - \frac{y p_x}{m}$$

$$\begin{aligned} \Rightarrow \{H, A_x\} &= \frac{\alpha p_x}{m(x^2 + y^2)^{1/2}} - \frac{\alpha p_x}{m(x^2 + y^2)^{3/2}} (x^2 + y^2) \\ &= \frac{\alpha p_x}{m(x^2 + y^2)^{1/2}} - \frac{\alpha p_x}{m(x^2 + y^2)^{1/2}} \frac{(x^2 + y^2)}{(x^2 + y^2)} \\ &\stackrel{!}{=} 0 \end{aligned} \quad \text{3P.}$$

→ Aus Symmetriegründen auch $\{H, A_y\} = 0$

$$\Rightarrow \{H, \vec{A}\} = 0 \Leftrightarrow \frac{d\vec{A}}{dt} = 0 \quad \text{1P.}$$

Aufgabe 3:

Schraubenlinie \rightarrow Zylinder

a) (1) $x^2 + y^2 = R^2$; $R > 0$

(2) $z = \frac{h}{2\pi} \phi$; $h > 0$

$$b := \frac{h}{2\pi}$$

$$\underline{r}(\phi) = \begin{pmatrix} R \cos \phi \\ R \sin \phi \\ b \phi \end{pmatrix}$$

1 P.

1 P.

b)

$$\underline{\underline{L = \frac{m}{2} (R^2 + b^2) \dot{\phi}^2 - mgb\phi}}$$

1 P.

c)

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = 0$$

1 P.

$$\frac{\partial L}{\partial \phi} = -mgb ; \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = m(R^2 + b^2) \ddot{\phi}$$

$$\Rightarrow \underline{\underline{\ddot{\phi} = -\frac{gb}{R^2 + b^2}}}$$

1 P.

d)

$$\underline{\underline{H = \frac{p_\phi^2}{2m(R^2 + b^2)} + mgb\phi}}$$

1 P.

e)

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} ; \quad \dot{p}_\phi = -\frac{\partial H}{\partial \phi}$$

$$\Rightarrow \underline{\underline{\ddot{\phi} = -\frac{gb}{R^2 + b^2}}}$$

2 P.

8) 1. Integration nach t :

$$\dot{\phi} = -\frac{gb}{R^2 + b^2} t + C_1$$

2. Integration:

$$\underline{\underline{\phi(t) = -\frac{1}{2} \frac{gb}{R^2 + b^2} t^2 + C_1 t + C_2}} \quad 1P.$$

$$\Rightarrow C_1 = \dot{\phi}(t=0) = \omega_0$$

$$C_2 = \phi(t=0) = \phi_0$$

g) Skizze:



1P.

Aufgabe 4:

a) $S(r) = \alpha r$; $\alpha > 0$

$\underline{\underline{J}}$ ist diagonal in allen Koord.-Systemen, die im Schwerpunkt der Kugel verankert sind.

$$\underline{\underline{J_{\mu\nu}}} = \int dV' S(r') \{ (r')^2 \delta_{\mu\nu} - (r')_{\mu} (r')_{\nu} \} \quad 1P.$$

Kugelkoordin.:

$$x = r \sin \vartheta \cos \phi$$

$$y = r \sin \vartheta \sin \phi$$

$$z = r \cos \vartheta$$

$$dV = r^2 \sin \vartheta \, dr \, d\vartheta \, d\phi$$

$$J_{11} = J_{22} = J_{33} \equiv J$$

$$J = J_{33} = \int_V dV \rho(r) (x^2 + y^2)$$

$$= \alpha \int_0^R dr \int_0^\pi d\vartheta \int_0^{2\pi} d\phi \, r \{ r^2 \sin^2 \vartheta \cos^2 \phi + r^2 \sin^2 \vartheta \sin^2 \phi \}$$

$$= \alpha \int_0^R dr \int_0^\pi d\vartheta \int_0^{2\pi} d\phi \, r^5 \sin^3 \vartheta$$

$$= 2\pi \alpha \int_0^R dr \int_0^\pi d\vartheta \, r^5 \sin \vartheta \underbrace{\sin^2 \vartheta}_{(1 - \cos^2 \vartheta)}$$

$$= \frac{2\pi \alpha R^6}{6} \int_0^\pi d\vartheta \sin \vartheta (1 - \cos^2 \vartheta)$$

$$= \frac{\pi \alpha R^6}{3} \left[\int_0^\pi d\vartheta \sin \vartheta - \int_0^\pi d\vartheta \sin \vartheta \cos^2 \vartheta \right]$$

$= -\frac{1}{3} \cos^3 \vartheta$

$$= \frac{\pi \alpha R^6}{3} \left\{ \underbrace{\left[-\cos \vartheta \right]_0^\pi}_2 + \underbrace{\left[\frac{1}{2} \cos^3 \vartheta \right]_0^\pi}_{-\frac{2}{3}} \right\}$$

$$= \frac{\pi \alpha R^6}{3} \cdot \frac{4}{3} = \frac{4}{9} \pi \alpha R^6 \quad 2P.$$

$$M = \int_V dV \rho(r) = 4\pi \alpha \int_0^R r^3 dr \\ = 4\pi \alpha \left[\frac{1}{4} r^4 \right]_0^R = \pi \alpha R^4$$

$$\Rightarrow \alpha(M) = \frac{M}{\pi R^4} \quad 1P.$$

$$\underline{\underline{J = \frac{4}{9} MR^2}} \quad 1P.$$

$$b) T_{\text{rot}} = \frac{1}{2} \underline{\underline{\omega}} J \underline{\underline{\omega}} \quad 1P.$$

$$= \frac{1}{2} \sum_{\mu=1}^3 \sum_{\nu=1}^3 \omega_{\mu} J_{\mu\nu} \omega_{\nu}$$

$$= \frac{1}{2} J (\omega_1^2 + \omega_2^2 + \omega_3^2)$$

$$= \frac{1}{2} J \omega^2$$

$$| J = \frac{4}{9} MR^2$$

$$\Rightarrow T_{\text{rot}} = \frac{2}{9} M (\omega R)^2$$

Abrollbedingung: $v = \omega R$

$$\Rightarrow T_{\text{rot}} = \frac{2}{9} M v^2 \quad 1P.$$

c) Energieerhaltung:

$$E_{\text{ges}} = V_{\text{pot}} (h = h_0) = Mgh_0$$

Am untersten Punkt Geschw. maximal $\rightarrow v_{\text{max}}$
und dann:

$$E_{\text{ges}} = T_{\text{trans.}} + T_{\text{rot}} + V_{\text{pot}} (h = 0)$$

$$Mgh_0 = \frac{1}{2} M v_{\text{max}}^2 + \frac{2}{9} M v_{\text{max}}^2 + 0$$

$$gh_0 = \frac{13}{18} v_{\text{max}}^2$$

$$\Rightarrow \underline{\underline{v_{\text{max}} = \sqrt{\frac{18}{13} gh_0}}}$$

1P.

$$h_0 = l \sin \beta \Rightarrow v_{\text{max}} = \sqrt{\frac{18}{13} gl \sin \beta}$$