Analyt. Näherung für $t_{cor}(D)$ für Van der Pol-Oszillator

(mean-field Näherung, Pompluun et al., Europhys. Lett. 71, 366 (2005))

Näherung für $D \ll |\varepsilon| \omega_0$: $\langle x^2 \rangle$ klein

$$\varepsilon - x^2 \approx \varepsilon - \langle x^2 \rangle \equiv \tilde{\varepsilon} < 0$$

$$\Rightarrow \boxed{\begin{aligned} \dot{x} &= y \\ \dot{y} &= \tilde{\varepsilon} y - \omega_0^2 x + D\xi(t) \end{aligned}}$$

lin. stoch. Dgl. (multivariate stationäre Ornstein-Uhlenbeck-Prozess) $\langle \underline{x}_s \rangle = 0$

$$d\underline{x}_s = -\underline{\underline{A}}\,\underline{x}_s\,dt + \underline{\underline{B}}\,d\underline{W}(t) \qquad \underline{x}_s = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \underline{\underline{A}} = \begin{pmatrix} 0 & -1 \\ \omega_0^2 & -\tilde{\varepsilon} \end{pmatrix}$$

$$\underline{\underline{B}} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$$

Varianz-Matrix

$$\underline{\underline{\sigma}} = \langle \underline{x}_s(t) \otimes \underline{x}_s(t) \rangle = \frac{D^2}{-2\tilde{\varepsilon}} \begin{pmatrix} \frac{1}{\omega_0^2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \langle x^2 \rangle = \frac{D^2}{-2\tilde{\varepsilon}\omega_0^2} = \frac{D^2}{-2(\varepsilon - \langle x^2 \rangle)\omega_0^2}$$

selbstkonsistente Lösung von $\langle x^2 \rangle$ in Abh. von D

$$\langle x^2 \rangle^2 - \varepsilon \langle x^2 \rangle - \frac{D^2}{2\omega_0^2} = 0$$

$$\langle x^2 \rangle = \frac{\varepsilon}{2} \pm \frac{\varepsilon}{2}\sqrt{1 + \frac{2D^2}{\varepsilon^2 \omega_0^2}} \qquad (+) \text{ unphys. Lös., da } \langle x^2 \rangle > 0$$

$$\Rightarrow \boxed{\tilde{\varepsilon} = \varepsilon - \langle x^2 \rangle = \frac{\varepsilon}{2}\left(1 + \sqrt{1 + \frac{2D^2}{\varepsilon^2 \omega_0^2}}\right)}$$

Dämpfung $\tilde{\varepsilon}$ = Abstand zur Hopf-Bif. wächst mit Rauschintensität D!

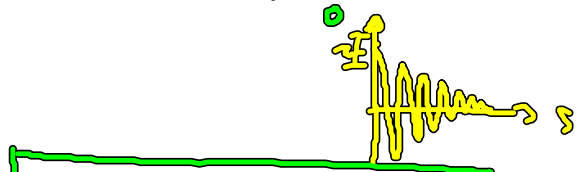
Autokorr.-fkt. für linearen stoch. Prozess

$$F_{yy}(s) = \langle y(t+s)\,y(t)\rangle \approx F_{yy}(0)\, e^{ps}\cos\tilde{\omega}s$$

with $\lambda_{1,2} = p \pm i\tilde{\omega} = \frac{\varepsilon}{2} \pm i\sqrt{\omega_0^2 - \frac{\varepsilon^2}{4}}$ die Eigenwerte von Ajacob.

$$\Rightarrow t_{cor} = \frac{1}{F(0)}\int_0^\infty |F(s)|\,ds \approx \frac{2}{\pi}\int_0^\infty e^{ps}\,ds = -\frac{2}{\pi p}$$

↑ Realteil



$$\Rightarrow \boxed{t_{cor}(D) \approx \frac{4}{\pi\,|\tilde{\varepsilon}(D)|}}$$

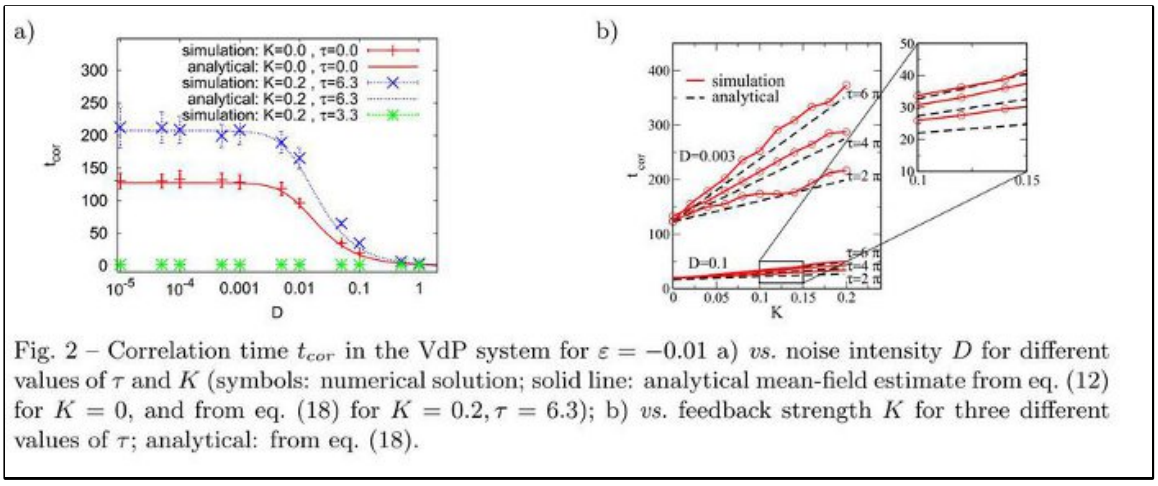


Fig. 2 – Correlation time t_{cor} in the VdP system for $\varepsilon = -0.01$ a) *vs.* noise intensity D for different values of τ and K (symbols: numerical solution; solid line: analytical mean-field estimate from eq. (12) for $K = 0$, and from eq. (18) for $K = 0.2, \tau = 6.3$); b) *vs.* feedback strength K for three different values of τ; analytical: from eq. (18).

Pomplun et al (2005)

Spektrale Leistungsdichte

Fourier-Trafo der mean-field-gl.: $\dot{x} = y$, $\dot{y} = \varepsilon y - \omega_0^2 x + D\xi(t)$

$\left(x(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\,\hat{x}(\omega)\right)$

$$-i\omega\hat{x}(\omega) = \hat{y}(\omega)$$
$$-i\omega\hat{y}(\omega) = \varepsilon\hat{y}(\omega) - \omega_0^2\hat{x}(\omega) + D\hat{\xi}(\omega)$$

Elim. von $\hat{x}(\omega) = \frac{i}{\omega}\hat{y}(\omega)$:

$$-i\omega\hat{y} - \varepsilon\hat{y} + i\frac{\omega_0^2}{\omega}\hat{y} = D\hat{\xi} \quad | \cdot i\omega$$

$$\hat{y}(\omega) = \frac{i\omega D\,\hat{\xi}(\omega)}{\omega^2 - \omega_0^2 - i\omega\tilde{\xi}}$$

$$\langle \hat{y}(\omega)\hat{y}^*(\omega')\rangle = \frac{(i\omega D)(-i\omega' D)\langle \hat{\xi}(\omega)\hat{\xi}^*(\omega')\rangle}{(\omega^2-\omega_0^2-i\omega\tilde{\xi})(\omega'^2-\omega_0^2+i\omega'\tilde{\xi})}$$

$$\langle \hat{\xi}(\omega)\hat{\xi}^*(\omega')\rangle = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\int_{-\infty}^{\infty} dt'\, e^{-i\omega' t'}\underbrace{\langle \xi(t)\xi(t')\rangle}_{\delta(t-t')}$$

$$= \frac{1}{2\pi}\cdot\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i(\omega-\omega')t}}_{\delta(\omega-\omega')}$$

R.S.: $\omega = \omega'$

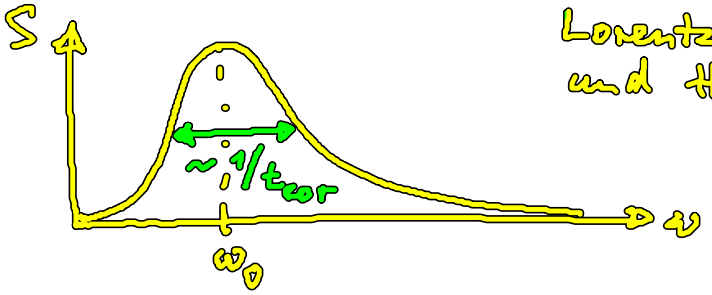
$$\Rightarrow \langle \hat{y}(\omega)\hat{y}^*(\omega')\rangle = \frac{D^2}{2\pi}\,\frac{\omega^2\delta(\omega-\omega')}{(\omega^2-\omega_0^2)^2+\omega^2\tilde{\xi}^2}$$

anderseits mit Wiener-Khinchin-Theorem:

$$\langle \hat{y}(\omega)\hat{y}^*(\omega')\rangle = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\int_{-\infty}^{\infty} dt'\, e^{-i\omega' t'}\langle y(t)y(t')\rangle \qquad t' = t+s$$

$$= \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i(\omega-\omega')t}}_{\delta(\omega-\omega')}\;\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} ds\, e^{i\omega' s}\langle y(t)y(t+s)\rangle}_{S(\omega')}$$

$$\boxed{S_{yy}(\omega) = \frac{D^2}{2\pi}\,\frac{\omega^2}{(\omega^2-\omega_0^2)^2+\omega^2\tilde{\xi}^2}}$$



Lorentzkurve mit Max. bei ω_0 und Halbwertsbreite $\approx \frac{|\tilde{\xi}|}{2} = \frac{2}{\pi t_{wr}(D)}$

Analyt. Näherung für FitzHugh-Nagumo-Modell:

Prager, Lerel, Schimansky-Geier, Schöll, J. Phys. A40, 11045 (2007)
Kovaric, Schimansky-Geier, Schöll: Eur. Phys. J-ST (2010/11)
2-state Master-Gl.

gekoppelte FitzHugh-Nagumo-Systeme mit Rauschen

Hauschildt, Balanov, Janson, Schöll: PRE 74, 051906 (2006)
Hövel, Dahlem, Schöll: Int. J. Bif. Chaos 20, 813 (2010)

stochastische Synchronisation

farbiges Rauschen: $\langle \eta(t)\eta(t')\rangle = \sigma^2 e^{-|t-t'|/\tau_c}$

$\Leftrightarrow$ generiert durch Ornstein-Uhlenbeck-Prozess

$$\tau_c \dot{\eta} = -\eta + \sqrt{2\sigma^2\tau_c}\,\xi(t) \qquad \langle \xi(t)\xi(t')\rangle = \delta(t-t')$$

Brandstetter, Dahlem, Schöll: Phil. Trans. R. Soc. A368, 391 (2010)

3. Beispiel: SNIPER-Modell (Anregbarkeit Typ I)

- knapp unterhalb einer globalen Bif. (Ampl. endl., $\omega \to 0$)
  Hopf: Ampl. $\to 0$, ω endl.

SNIPER $\triangleq$ saddle-node infinite period bif.
$\equiv$ SNIC $\triangleq$ saddle-node bif. on invariant cycle
$\equiv$ Sattel-Knoten-Bif. auf Grenzzyklus

Gang et al.: PRL 71, 807 (1993): Kohärenzresonanz erstmals
Aust, Hövel, Rizanides, Schöll: Eur. Phys. J-ST 187, 77 (2010) mit delay

$$\dot{x} = x(1-x^2-y^2) + y(x-b) + D\xi(t)$$
$$\dot{y} = y(1-x^2-y^2) - x(x-b) + D\xi(t)$$

$D=0$

$$\dot{r} = r(1-r^2)$$
$$\dot{\varphi} = b - r\cos\varphi$$

$x = r\cos\varphi$
$y = r\sin\varphi$

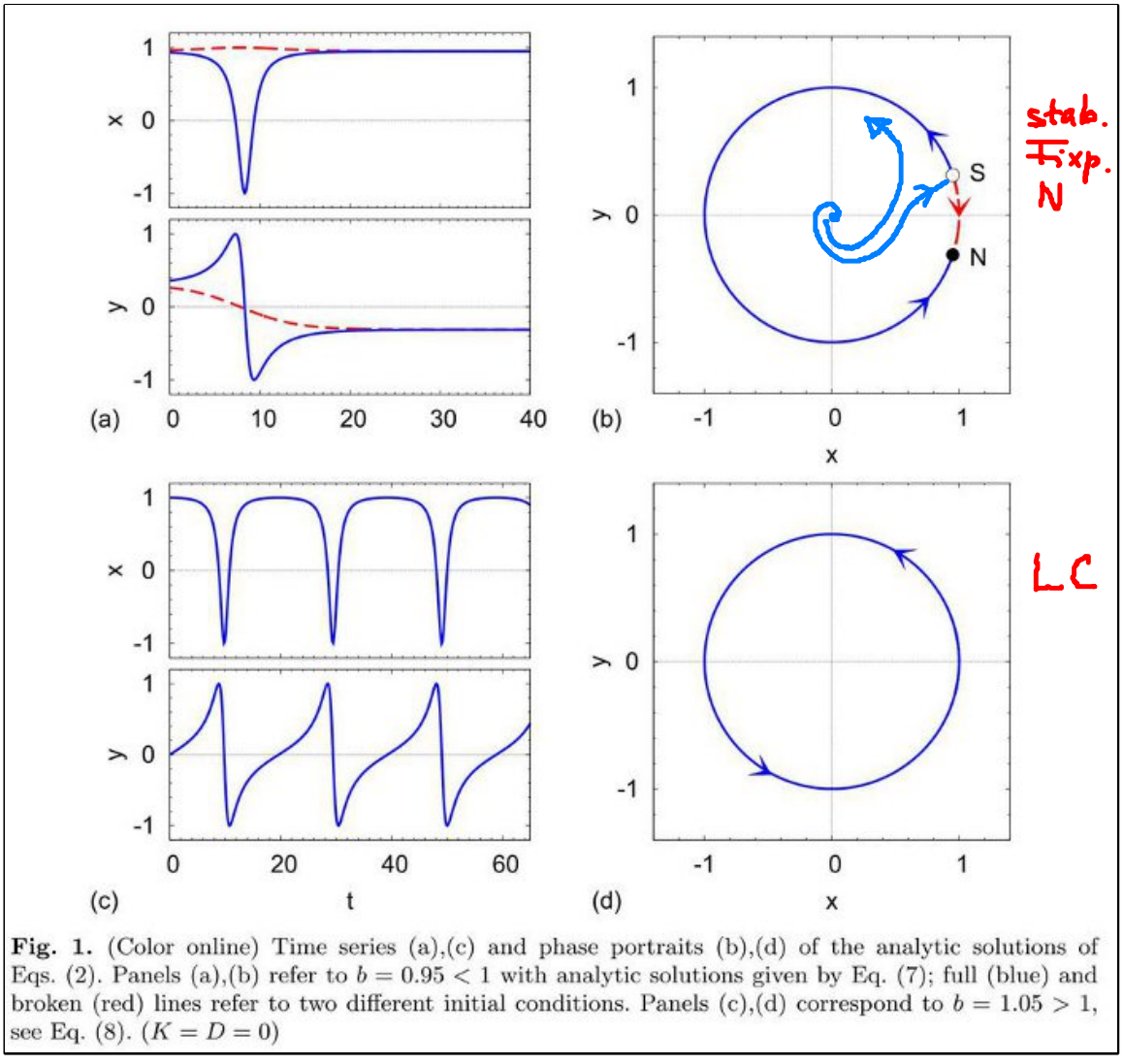
Fixp.: (i) $r=0$ (instab. Fokus)
(ii) $r=1$, $\varphi = \arccos b$ (Sattelp. S)
(iii) $r=1$, $\varphi = -\arccos b$ (stab. Knoten N)
} ex. falls $b<1$

$b = 1$: Sattel-Knoten-Bif. auf dem invarianten Kreis $r = 1$
($r = 1$, $\dot{r} = 0$, $\dot{\varphi} = b - \cos\varphi$) bei $\varphi = 0$

$b > 1$: Grenzzyklus $r = 1$, $\int \frac{d\varphi}{b - \cos\varphi} = t \Rightarrow \varphi(t)$ analyt.

$T = \frac{2\pi}{\sqrt{b^2 - 1}} \xrightarrow{b \to 1} \infty$

Anregbare Regime : $b < 1$



stab. Fixp. N

Schwelle S (für Anregung)

Arnol'd et al (2010)

LC

Fig. 1. (Color online) Time series (a),(c) and phase portraits (b),(d) of the analytic solutions of Eqs. (2). Panels (a),(b) refer to $b = 0.95 < 1$ with analytic solutions given by Eq. (7); full (blue) and broken (red) lines refer to two different initial conditions. Panels (c),(d) correspond to $b = 1.05 > 1$, see Eq. (8). ($K = D = 0$)

$D \neq 0$: Kohärenzresonanz

Arnold et al (2010

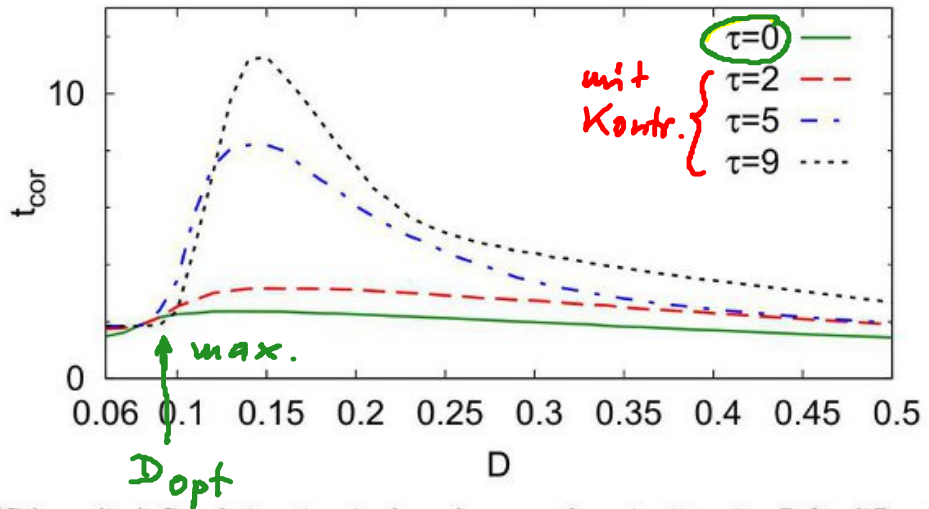


Fig. 4. (Color online) Correlation time in dependence on the noise intensity D for different time delays τ. The solid (green) curve corresponds to the uncontrolled system ($\tau = 0$). The dashed (red), dash-dotted (blue), and dotted (black) curves refer to values of $\tau = 2, 5$, and 9, respectively. Other parameters: $b = 0.95$ and $K = 0.25$.

Gang et al PRL (1993)

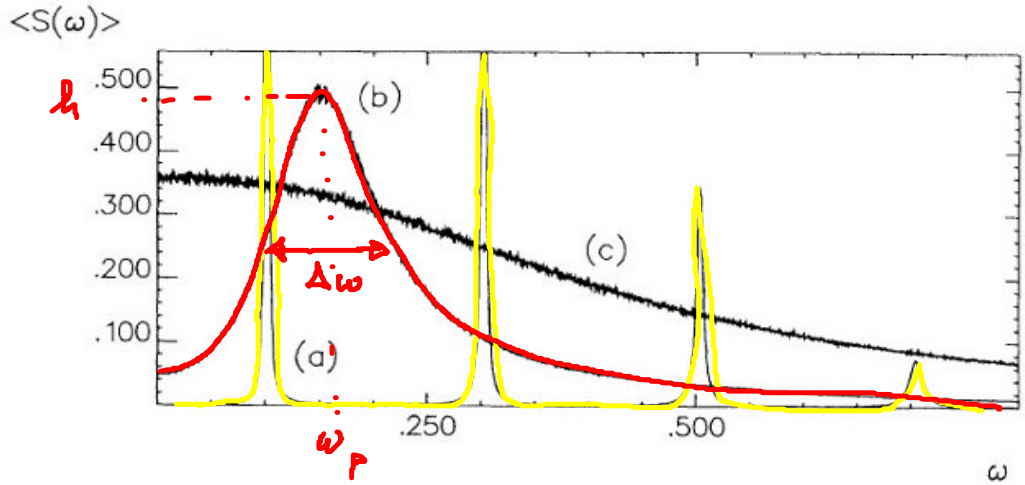


FIG. 1. The averaged power spectrum of $y(t)$ for $b = 1.05$ and various D. A limit cycle exists for the deterministic system. (a) $D = 0.00003$. (b) $D = 0.05$. (c) $D = 0.9$.

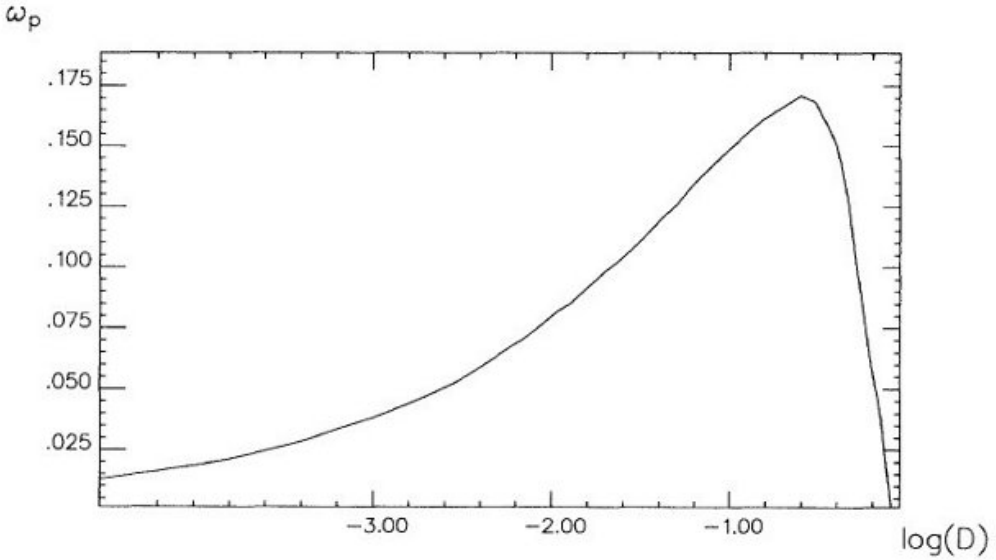


FIG. 3. The center frequency of the spectrum peak ω_p vs $\log(D)$ with $b=1$ (the same in Figs. 4 and 5).

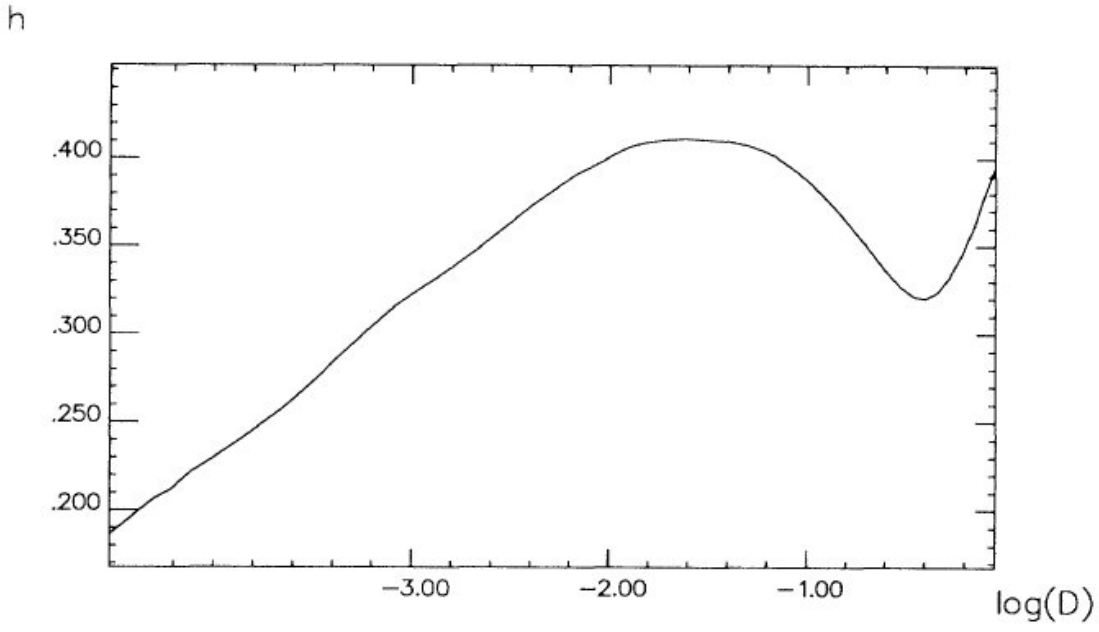


FIG. 4. The spectrum peak height h plotted against $\log(D)$. A resonancelike curve is obvious. However, h goes up again for very large D.

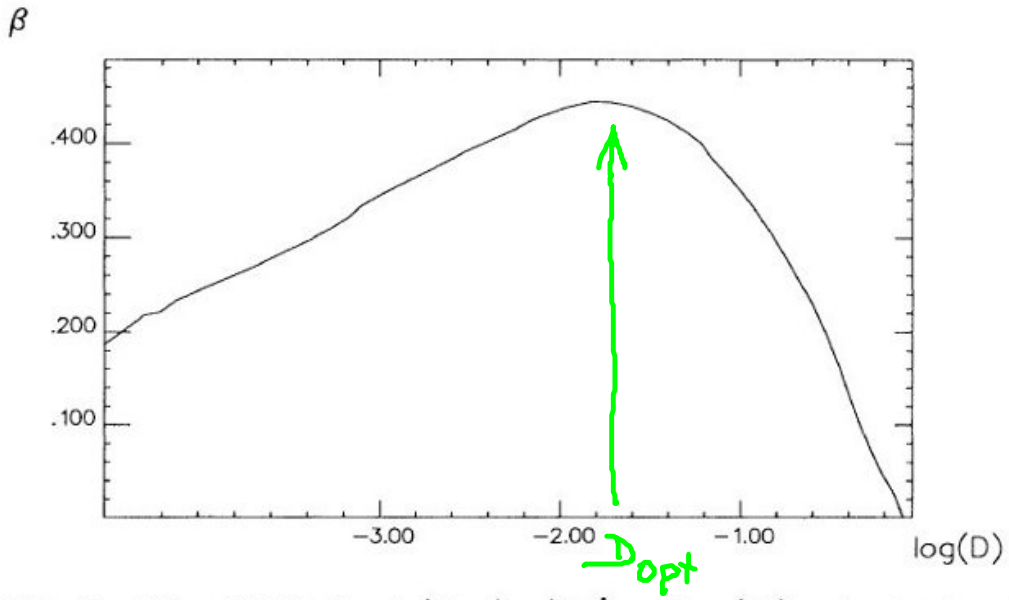


FIG. 5. The SNR $\beta = h(\Delta\omega/\omega_p)^{-1}$ vs $\log(D)$. A stochastic resonance maximum can be seen.