

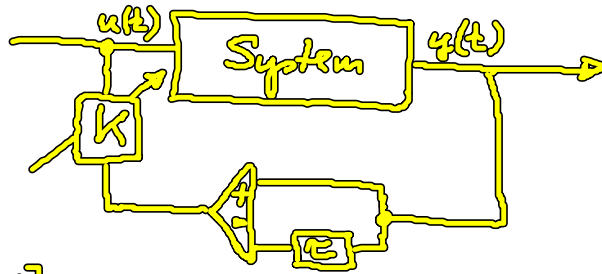
3.2.1 Stabilisierung instabiler Fixpunkte durch zeitverzögerte Rückkopplungskontrolle

Pyragas (1992)
 Phys. Lett. A 170, 421

$y(t)$: Ausgangsvar.

$$u(t) = K[y(t) - y(t-\tau)]$$

Kontrollvar.



closed-loop control
 (Rückkoppl. Kontrolle)

Verzög.-zeit τ
 Rückkoppl.-stärke K

- nichtinvasiv

(Kontrollkraft verschwindet auf dem Zielzustand $y(t) = y(t-\tau)$)

Allg. Form eines 2-Var.-System (ohne Kontrolle)

Fixpkt. \underline{x}^* : $0 \stackrel{!}{=} \dot{\underline{x}} = f(\underline{x}^*) \quad \underline{x} \in \mathbb{R}^2$

Linearisierung um \underline{x}^* für kleine Störungen:

$$\underline{x}(t) = \underline{x}^* + \delta \underline{x}(t) : \delta \dot{\underline{x}} = (Df)_{\#} \delta \underline{x}$$

Lös. $\delta \underline{x} \sim e^{\lambda t}$: $0 = \det(\lambda I - A) = \lambda^2 - \lambda \operatorname{tr} A + \det A$

Jacobi-Matrix $(Df)_{\#} \equiv A$

$$\Rightarrow \lambda = \frac{\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4 \det A}}{2}$$

Normalform eines instabilen Fokus $\lambda = \alpha \pm i\omega \quad (\alpha > 0)$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

komplexe Schreibweise $\dot{z} = (\alpha \pm i\omega)z$, $z = x \mp iy \in \mathbb{C}$

mit zeitverzögerter Rückkopplung:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda & \omega \\ -\omega & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - K \begin{pmatrix} x(t) - x(t-\tau) \\ y(t) - y(t-\tau) \end{pmatrix}$$

"diagonale Rückkopplung"

$$\dot{z} = (\lambda \pm i\omega)z - K(z(t) - z(t-\tau))$$

Ansatz: $z(t) = e^{\lambda t}$

$$\text{char. Gl. } 0 = \det \left[\begin{pmatrix} \lambda - \lambda & \omega \\ -\omega & \lambda - \lambda \end{pmatrix} - K \begin{pmatrix} 1 - e^{-\lambda\tau} & 0 \\ 0 & 1 - e^{-\lambda\tau} \end{pmatrix} \right]$$

$$= [\lambda + K(1 - e^{-\lambda\tau}) - \lambda]^2 + \omega^2 \Leftrightarrow []^2 = -\omega^2$$

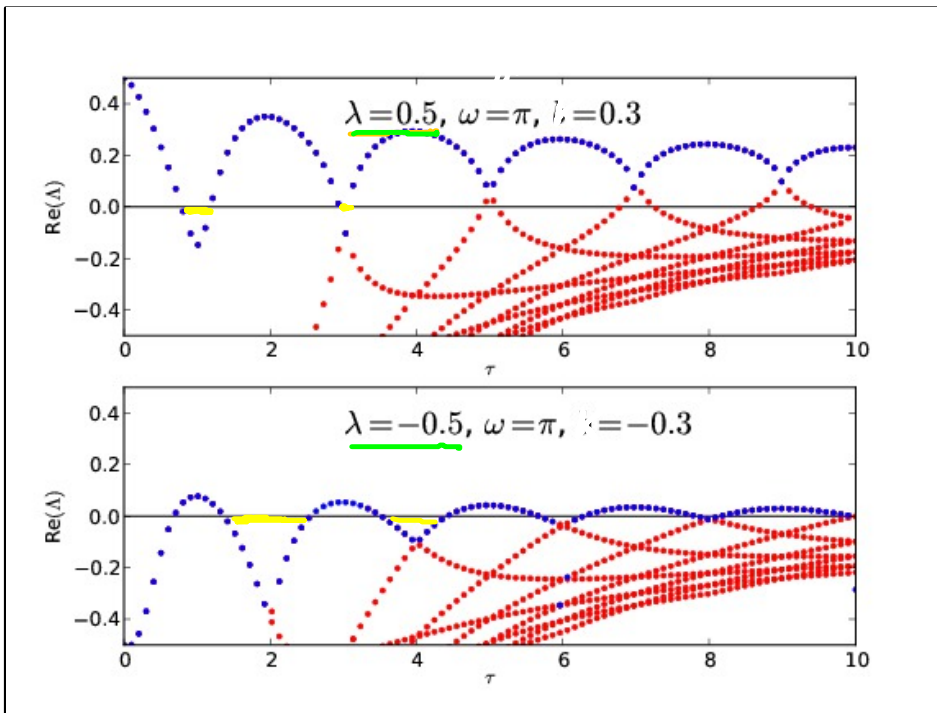
$$\Rightarrow \lambda + K(1 - e^{-\lambda\tau}) = \lambda \pm i\omega$$

Lösung durch Lambertfkt. $(\lambda + K - (\lambda \pm i\omega))\tau = K\tau e^{-\lambda\tau}$

$$z e^z = K\tau e^{-(\lambda \pm i\omega)\tau + K\tau}$$

$$\lambda\tau = W(K\tau e^{-(\lambda \pm i\omega)\tau + K\tau}) + (\lambda \pm i\omega)\tau - K\tau$$

natürliche Zeitskala: $T_0 = \frac{2\pi}{\omega}$ (Osz. periode ohne Delay)

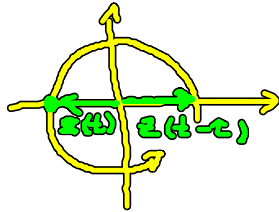


unkontrolliert:
Fixpt. instabil
($\lambda > 0$)

unkontrolliert:
Fixpt. stabil
($\lambda < 0$)

nichtmonotones Verhalten der Eigenmoden
 $\operatorname{Re} \lambda$ als Fkt. von τ führt abwechselnd
 zu Stabilisierung / Destabilisierung

Stabilitätsumkehr bei $\tau = \frac{2n+1}{2} T_0$ ($n=0,1,2,\dots$)



Stabilitätsgrenzen: $\operatorname{Re} \lambda = 0$

Char. gl. in Re und Im aufgespalten mit $\lambda = p + iq$:

$$\lambda = p + K [1 - e^{-p\tau} \cos q\tau] \quad a > 0$$

$$\omega = q + K e^{-p\tau} \sin q\tau$$

↓

$$\operatorname{Re} \lambda = p = 0: \quad \lambda = K(1 - \cos q\tau) \quad (1)$$

$$\omega = q + K \sin q\tau \quad (2)$$

Systempar. λ, ω geg.
 Kontrollpar. $K, \tau > 0$

Kurvenpar. der Stab.grenze im (K, τ) -Raum: q

$$(1) \Rightarrow 0 \leq 1 - \cos q\tau \leq 2 \Rightarrow \frac{2}{K} \leq 2 \Rightarrow \boxed{K \geq \frac{2}{2}}$$

Notwendige Bed. für Stabilität.

(min. Rückkoppl.stärke)

$$K_{\min} = \frac{2}{2} \stackrel{(1)}{\Rightarrow} \cos q\tau = -1 \Rightarrow q\tau = (2n+1)\pi \quad (3)$$

$n=0,1,2,\dots$

$$\Rightarrow \sin q\tau = 0 \stackrel{(2)}{\Rightarrow} q = \omega$$

$$\Rightarrow \boxed{\tau = \frac{\pi}{\omega} (2n+1) = T_0 \frac{2n+1}{2}} \quad n=0,1,2,\dots$$

Für $\tau = \frac{2\pi n}{\omega} = nT_0$ ist keine Stabilisierung möglich, weil

$$(2) \Rightarrow q = \omega \stackrel{(1)}{\Rightarrow} \frac{k-2}{k} = \cos(q\tau)_{q\tau=2\pi n} = 1 \Leftrightarrow 1 - \frac{2}{k} = 1$$

Analyt. Berechnung der Stabilitätsgrenze in der (K, τ) -Ebene:

$$(1), (2) \Rightarrow \left(\frac{k-2}{k}\right)^2 + \left(\frac{\omega-q}{k}\right)^2 = \cos^2 q\tau + \sin^2 q\tau = 1$$

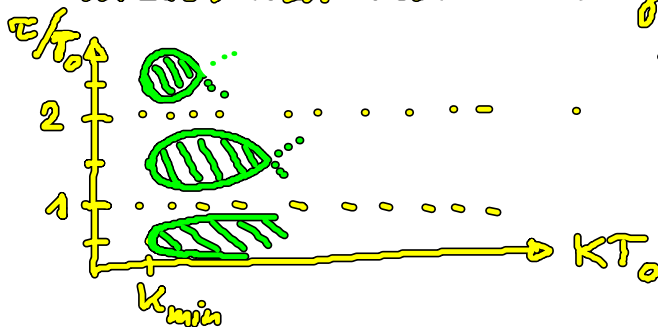
$$\Rightarrow \omega - q = \pm k \sqrt{1 - \left(\frac{k-2}{k}\right)^2}$$

$$q = \omega \mp \sqrt{(2k-2)\lambda} \quad \text{eliminiert } q \text{ aus (1), (2)}$$

fixe Par. λ, ω des unkontrollierten Systems

$$\Rightarrow \text{Relation zwischen } \tau \text{ und } k \text{ aus (1): } \tau(k) = \frac{\arccos \frac{k-2}{k}}{\omega \mp \sqrt{(2k-2)\lambda}}$$

\arccos hat mehrere Zweige:



Stabilisierungszonen!

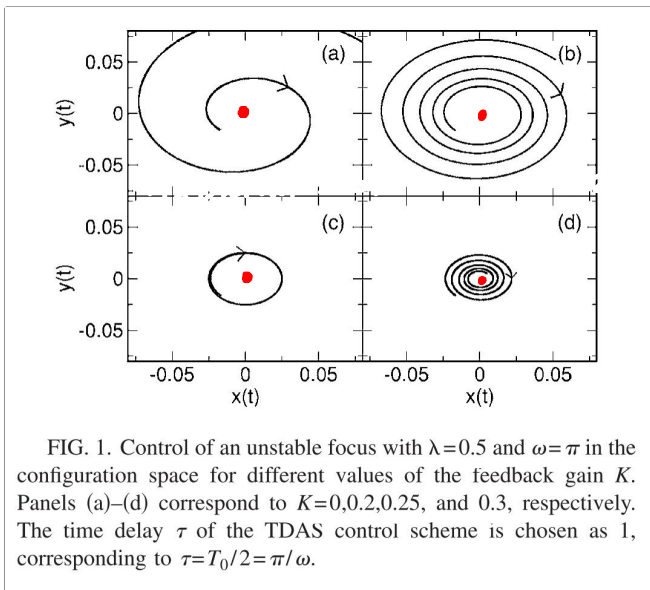


FIG. 1. Control of an unstable focus with $\lambda=0.5$ and $\omega=\pi$ in the configuration space for different values of the feedback gain K . Panels (a)–(d) correspond to $K=0, 0.2, 0.25$, and 0.3 , respectively. The time delay τ of the TDAS control scheme is chosen as 1, corresponding to $\tau=T_0/2=\pi/\omega$.

Hövel u. Schöll:

Phys. Rev. E 72, 04620 (2005)

Stabilisierung für geeignete τ und K !

$$\tau \approx \frac{T_0}{2}, \frac{3T_0}{2}, \dots$$

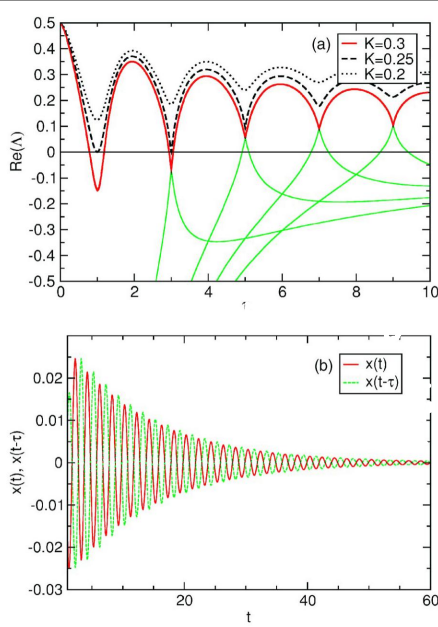


FIG. 2. (Color online) (a) Largest real part of the complex eigenvalues Λ vs τ for $\lambda=0.5$ and $\omega=\pi$ for different K . Some lower eigenvalues are also displayed for $K=0.3$ (green online). (b) Time series of the x component of the unstable focus: The solid line (red online) corresponds to $x(t)$, the dashed line (green online) to the delayed x component $x(t-\tau)$ with $\tau=1$. The parameters of the unstable focus and the control scheme are as in panel (d) of Fig. 1.

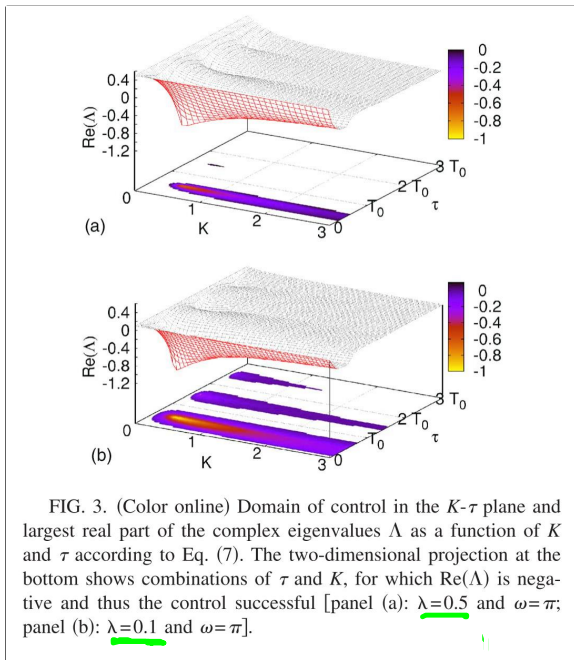
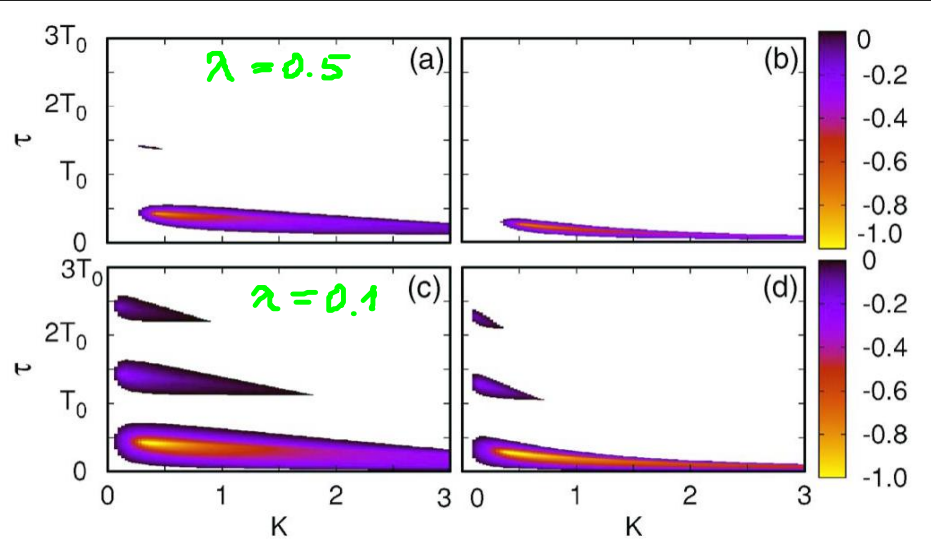


FIG. 3. (Color online) Domain of control in the K - τ plane and largest real part of the complex eigenvalues Λ as a function of K and τ according to Eq. (7). The two-dimensional projection at the bottom shows combinations of τ and K , for which $\text{Re}(\Lambda)$ is negative and thus the control successful [panel (a): $\lambda=0.5$ and $\omega=\pi$; panel (b): $\lambda=0.1$ and $\omega=\pi$].

$$\lambda = 0.5$$

$$\lambda = 0.1$$



Latenzzeit
 $x(t-\delta) - x(t-\delta-\tau)$

FIG. 6. (Color online) Domain of control in the K - τ plane for different latency times [panels (a) and (c): $\delta=0.1$; panels (b) and (d): $\delta=0.3$]. The shaded areas indicate combinations of τ and K , for which the largest real part of the complex eigenvalues Λ is negative and thus control is successful. The value of $\text{Re}(\Lambda)$ is indicated by the greyscale (color online). The parameters of the unstable focus are chosen as $\omega=\pi$ in all panels and $\lambda=0.5$ in (a) and (b) and $\lambda=0.1$ in (c) and (d).

Erweiterung (Socolar, Sukov, Gauthier, PRL 50, 3245(1993))

- multiple-time feedback (ETDAS, extended time-delay autosynchronization)

$$K \sum_{n=0}^{\infty} R^n [x(t-n\tau) - x(t-(n+1)\tau)]$$
 Gradientenpar.R
 $(0 \leq R < 1)$

Eigenwertgl.

$$\Lambda + K \frac{1 - e^{-\lambda\tau}}{1 - R e^{-\lambda\tau}} = \lambda \pm i\omega$$

Stab. Bereich vergrößert

Dahms, Hövel, Schöll, PRL 76, 056201 (2007)

- Latenz-Effekte
 $K = [x(t-\delta) - x(t-\delta-\tau)]$ Latenzzeit δ
 Stab. Bereich verkleinert
- phasen-abhängige Rückkoppl.

$$K \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x(t) - x(t-\tau) \\ y(t) - y(t-\tau) \end{pmatrix}$$

- asymptot. Skalierungsverhalten für große τ
Yanchuk, Wolfram, Hövel, Schölk, PRE 74, 026201 (2006)
Hövel, PhDthesis (TU Berlin 2009; Springer 2010)
Wolfram et al, EPL-ST (2010)