

• Drehimpuls:

$$\underline{L}_I = \underline{L}_S + \underline{L}$$

mit $\underline{L} = \sum_v m_v \underline{r}_v \times (\underline{\omega} \times \underline{r}_v)$... bezogen auf Aufpkt. R

$$\rightarrow \boxed{\underline{L} = \underline{\Theta} \underline{\omega} \quad \text{mit} \quad \underline{\Theta}(t) = \sum_v m_v [|\underline{r}_v|^2 \underline{1} - \underline{r}_v(t) \otimes \underline{r}_v(t)]} \quad (10.24)$$

... Trägheitstensor in coord.
unabhängiger Form

Komponentendarstellung:

$$\boxed{L_i = \Theta_{ij} \omega_j \quad \text{mit} \quad \Theta_{ij} = \underline{e}_i \cdot \underline{\Theta} \underline{e}_j = \sum_v m_v [|\underline{r}_v|^2 \delta_{ij} - x_{vi} x_{vj}]} \quad (10.25)$$

• symmetr. Tensor 2.St.: $\Theta_{ij} = \Theta_{ji} \rightarrow 6$ unabh. Elemente

Hauptdiagonale: $\Theta_{11}, \Theta_{22}, \Theta_{33}$... Trägheitsmomente

Nebendiagonale: $\Theta_{12}, \Theta_{23}, \Theta_{13}$... Deviationsmomente

\rightarrow Lager kräfte Bsp. Rad

• Trägheitsmoment bzgl. Achse \hat{v} mit $|\hat{v}| = 1$:

$$\Theta_{vv} = \hat{v} \cdot \underline{\Theta} \hat{v} \quad (10.26)$$

• $\underline{\Theta} = \underline{\Theta}(t)$. Wo steckt Zeitabhängigkeit relativ zu IS?

(i) ONB des KS: $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$... körperfest!

$$\underline{\Theta}(t) = \Theta_{ij} \underline{e}_i(t) \otimes \underline{e}_j(t), \quad \Theta_{ij} = \underline{e}_i(t) \cdot \underline{\Theta} \underline{e}_j(t) \quad (10.27)$$

... zeitunabhängig

(ii) ONB des IS: $\{\underline{e}_{I1}, \underline{e}_{I2}, \underline{e}_{I3}\}$... raumfest

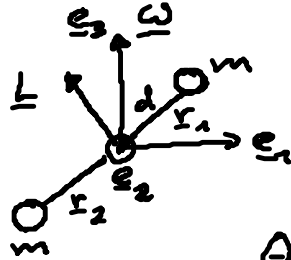
$$\underline{\Theta}(t) = \Theta_{Iij}(t) \underline{e}_{Ii} \otimes \underline{e}_{Ij}, \quad \Theta_{Iij}(t) = \underline{e}_{Ii} \cdot \underline{\Theta}(t) \underline{e}_{Ij} \quad (10.28)$$

... zeitabh.

• kontinuierliche Massenverteilung:

$$\underline{\underline{\Theta}} \stackrel{(10.9)}{=} \int d^3r \underbrace{\rho(\underline{r})}_{dm} [|\underline{r}|^2 \underline{\underline{1}} - \underline{r} \otimes \underline{r}] \quad (10.29)$$

• Bsp:
◀ "Hantel"



$$\underline{r}_1 = \frac{d}{\sqrt{2}} (\underline{e}_1 + \underline{e}_3), \quad |\underline{r}_1| = d$$

$$\underline{r}_2 = -\frac{d}{\sqrt{2}} (\underline{e}_1 + \underline{e}_3)$$

$$\Theta_{11} = \Theta_{33} = md^2 \cdot 2 \left(1 - \frac{1}{2}\right) = md^2$$

$$\Theta_{13} = \Theta_{31} = md^2 \cdot 2 \left(-\frac{1}{2}\right) = -md^2$$

$$\Theta_{22} = md^2 \cdot 2 \cdot 1 = 2md^2$$

$$\Theta_{ij} = 0, \text{ sonst}$$

$$\rightarrow \underline{\underline{\Theta}} = md^2 \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad (10.30)$$

$$\underline{\omega} = \omega \underline{e}_3: \quad \underline{L} = \underline{\underline{\Theta}} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = md^2 \omega \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{L} \neq \underline{H} \underline{\omega} \rightarrow \frac{d\underline{L}}{dt} \neq 0 \iff \text{Lagerkräfte}$$

e) kinetische Energie: $T = \frac{1}{2} \sum_{\nu} m_{\nu} \dot{\underline{r}}_{I\nu}^2$
mit $\dot{\underline{r}}_{I\nu} = \underline{\dot{R}} + \underline{\omega} \times \underline{r}_{\nu}$

(i) $\underline{R} = \underline{R}_S: \quad \sum_{\nu} m_{\nu} \underline{r}_{\nu} = 0$

$$T = T_{\text{trans}} + T_{\text{rot}} = \frac{1}{2} M \dot{\underline{R}}_S^2 + \frac{1}{2} \sum_{\nu} m_{\nu} (\underline{\omega} \times \underline{r}_{\nu})^2 \quad (10.31)$$

Untersuche: $\underline{\omega} \cdot \underline{L} \stackrel{(10.29)}{=} \sum_{\nu} m_{\nu} \underbrace{\underline{\omega} \cdot [\underline{r}_{\nu} \times (\underline{\omega} \times \underline{r}_{\nu})]}_{\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{c} \cdot (\underline{a} \times \underline{b})}$

$$= \sum_{\nu} m_{\nu} (\underline{\omega} \times \underline{r}_{\nu})^2$$

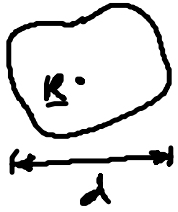
$$\rightarrow \underline{T}_{\text{rot}} = \frac{1}{2} \underline{\omega} \cdot \underline{L} = \frac{1}{2} \underline{\omega} \cdot \underline{\underline{\Theta}} \underline{\omega} = \frac{1}{2} \omega_i \Theta_{ij} \omega_j \quad (10.32)$$

... für Rotation um Aufpkt.

$$[\text{vgl. } T_{\text{trans}} = \frac{1}{2} m v^2]$$

$$(ii) \dot{\underline{R}} = 0: \quad \boxed{T = T_{\text{rot}}} \quad (10.33)$$

f) potenzielle Energie:



$U(\underline{r}_{Iv})$ für m_v , $v = 1, \dots, N$

$$\underline{r}_{Iv} = \underline{R} + \underline{r}_v \quad \text{mit } |\underline{r}_v| \leq d$$

$$U(\underline{r}_{Iv}) \stackrel{\text{Taylor}}{=} U(\underline{R}) + \underline{r}_v \cdot \text{grad}_v U(\underline{r}_{Iv}) \Big|_{\underline{R}}$$

$$\text{falls: } d |\text{grad}_v U(\underline{r}_{Iv}) \Big|_{\underline{R}}| \ll U(\underline{R})$$

$$\rightarrow \boxed{U(\underline{r}_{Iv}) \approx U(\underline{R})} \quad (10.34)$$

g) Eigenschaften von $\underline{\Theta}$:

(1) Hauptachseninfo.

$$\begin{aligned} \underline{\Theta} \underline{e}^{(i)} &= \Theta_i \underline{e}^{(i)} \\ \Theta_i &\geq 0 \quad \dots \text{Hauptträgheitsmomente} \\ \{ \underline{e}^{(1)}, \underline{e}^{(2)}, \underline{e}^{(3)} \} &\dots \text{Hauptachsensystem} \end{aligned} \quad \left. \vphantom{\begin{aligned} \underline{\Theta} \underline{e}^{(i)} &= \Theta_i \underline{e}^{(i)} \\ \Theta_i &\geq 0 \dots \text{Hauptträgheitsmomente} \\ \{ \underline{e}^{(1)}, \underline{e}^{(2)}, \underline{e}^{(3)} \} &\dots \text{Hauptachsensystem} \end{aligned}} \right\} \text{ von } \underline{\Theta} \quad (10.35)$$

$$\text{NB: kinet. Energie} \geq 0 \rightarrow \Theta_i \geq 0$$

$$\text{mit } \boxed{\underline{e}^{(i)} \cdot \underline{e}^{(j)} = \delta_{ij}}$$

• Diagonalgestalt von $\underline{\Theta}$:

$$\tilde{\Theta}_{ij} = \underline{e}^{(i)} \cdot \underline{\Theta} \underline{e}^{(j)} = \Theta_j \underline{e}^{(i)} \cdot \underline{e}^{(j)}$$

$$= \Theta_j \delta_{ij} \rightarrow \tilde{\underline{\Theta}} = \begin{pmatrix} \Theta_1 & 0 & 0 \\ 0 & \Theta_2 & 0 \\ 0 & 0 & \Theta_3 \end{pmatrix} \quad (10.36)$$

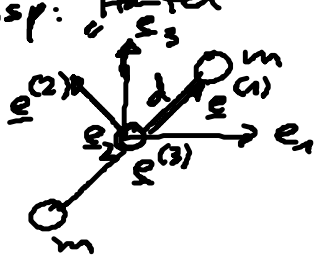
keine Summe!

$$\text{also: } \underline{\Theta} = \sum_{i=1}^3 \Theta_i \underline{e}^{(i)} \otimes \underline{e}^{(i)} \quad (10.37)$$

3 EW \rightarrow 3 Eulerwinkel \rightarrow 6 unabh. Komp von $\underline{\Theta}$

- Fälle: (i) $\Theta_1 \neq \Theta_2 \neq \Theta_3$: ^{allgemeiner} unsymmetr. Kreisel
- (ii) $\Theta_1 = \Theta_2 \neq \Theta_3$: achsen " " ("Rotationsellipsoid")
- (iii) $\Theta_1 = \Theta_2 = \Theta_3$: Würfel oder Kugel
- physikal: $\underline{\omega} \parallel \underline{e}^{(i)} \rightarrow \underline{L} \parallel \underline{\omega} \dots$ stabile Drehrichtungen!

• Bsp: "Hantel"



$$\underline{\underline{\Theta}} = md^2 \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

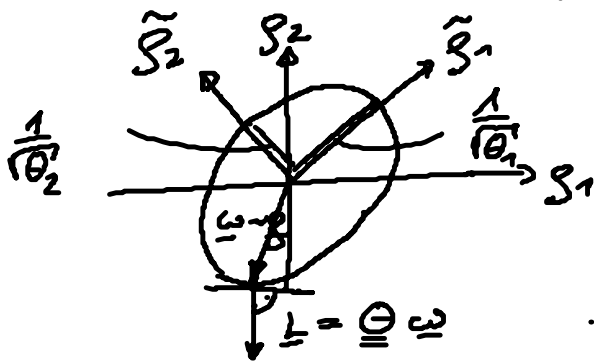
$$\left. \begin{aligned} \underline{e}^{(1)} &= \frac{1}{\sqrt{2}} (\underline{e}_1 + \underline{e}_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \Theta_1 = 0 \\ \underline{e}^{(2)} &= \frac{1}{\sqrt{2}} (-\underline{e}_1 + \underline{e}_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \Theta_2 = 2md^2 \\ \underline{e}^{(3)} &= \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \Theta_3 = 2md^2 \end{aligned} \right\} \underline{\underline{\tilde{\Theta}}} = 2md^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) Trägheitsellipsoid = Fläche konst. Rot. energie

• $T = \frac{1}{2} \underline{\omega} \cdot \underline{\underline{\Theta}} \underline{\omega}$ mit $\underline{s} = \frac{\underline{\omega}}{\sqrt{2T}}$ (10.38)

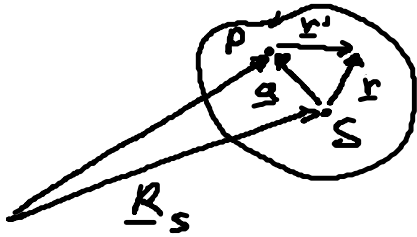
$$\rightarrow \boxed{1 = \underline{s}_i \Theta_{ij} \underline{s}_j \xrightarrow[\text{achsentransf.}]{\text{Haupt-}} 1 = \Theta_1 \tilde{s}_1^2 + \Theta_2 \tilde{s}_2^2 + \Theta_3 \tilde{s}_3^2}$$

... Trägheitsellipsoid mit Halbachsen $\frac{1}{\sqrt{\Theta_i}}$
(Fläche 2. Grades)



... Poinsot'sche Konstruktion
(Beweis: Übungen)

(3) Satz von Steiner:



$\underline{\Theta}^{(S)}$ bezogen auf S bekannt
 R Schwerpunkt

$$\rightarrow \underline{\Theta}^{(P)} = M(|a|^2 \underline{1} - \underline{a} \otimes \underline{a}) + \underline{\Theta}^{(S)} \quad (10.35)$$

Gesamtmasse ... Trägheits tensor bzgl. Punkt P

Beweis: Übung

10.3 Dynamik des starren Körpers (II): Eulersche Gleichungen

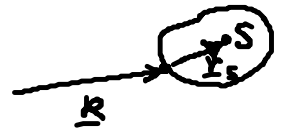
a) Dynam. Grundgleichungen

• Impulssatz:

$$\begin{matrix} (10.15) \\ (8.8) \end{matrix} \quad \underline{\dot{p}} = \sum_v \underline{F}_v^{(a)} = \underline{F}^{(a)}$$

(i) $\underline{R} = \underline{R}_S$: $\underline{p} = M \underline{\dot{r}}_S$ (10.16)

(ii) $\underline{\dot{r}} = 0$: $\underline{p} = \underline{\omega} \times M \underline{r}_S$ (10.17)



• Drehimpulssatz:

$$(10.14) \quad \dot{\underline{L}}_{\underline{I}} = \underline{D}_{\underline{I}}^{(a)} = \sum_{\nu} \underline{r}_{\underline{I}\nu} \times \underline{F}_{\nu}^{(a)} \quad \underline{R} \times \underline{F}^{(a)} + \underline{D} \quad (10.40)$$

(8.14)

$$\underline{r}_{\underline{I}\nu} = \underline{R} + \underline{r}_{\nu}$$

$$\text{mit } \underline{D} = \sum_{\nu} \underline{r}_{\nu} \times \underline{F}_{\nu}^{(a)} \quad (10.41)$$

... Drehmoment bzgl. Aufpkt. \underline{R}

$$(10.18) \quad \dot{\underline{L}} = \underline{L}_{\underline{S}} + \underline{L}$$

(10.21)

$$(i) \quad \underline{R} = \underline{R}_{\underline{S}} : \quad \dot{\underline{L}}_{\underline{S}} = (\underline{R}_{\underline{S}} \times \underbrace{M \dot{\underline{R}}_{\underline{S}}}_{\underline{p}}) \dot{\quad} = \underline{R}_{\underline{S}} \times \dot{\underline{p}} = \underline{R}_{\underline{S}} \times \underline{F}^{(a)} \quad (10.42)$$

$\underline{R}_{\underline{S}} \times \dot{\underline{R}}_{\underline{S}} = 0$

$$(ii) \quad \dot{\underline{R}} = 0 : \quad \dot{\underline{L}}_{\underline{S}} = \underline{R} \times \dot{\underline{p}} = \underline{R} \times \underline{F}^{(a)} \quad (10.43)$$

$$\Rightarrow \quad \underline{\dot{L}} = \underline{D}$$

... Drehimpulssatz für starren Körper
bzgl. Aufpkt. $\underline{R} \rightarrow$ Rotationsbewegung

$$\left[\text{vgl. Newton: } \dot{\underline{p}} = \underline{F} \text{ mit } \underline{p} = m \underline{v} \leftrightarrow \underline{L} = \underline{Q} \underline{\omega} \right]$$

$$\underline{F} \leftrightarrow \underline{D}$$