· Drehimpuls:

$$\underline{L}_I = \underline{L}_S + \underline{L}$$

mit $\underline{L} = \sum_\nu m_\nu \underline{r}_\nu \times (\underline{\omega} \times \underline{r}_\nu)$ ... bezogen auf Aufpkt. $\underline{R}$

$$\longrightarrow \boxed{\underline{L} = \underline{\underline{\Theta}}\,\underline{\omega} \quad \text{mit} \quad \underline{\underline{\Theta}}(t) = \sum_\nu m_\nu \left[|\underline{r}_\nu|^2 \mathbb{1} - \underline{r}_\nu(t) \otimes \underline{r}_\nu(t) \right]} \quad (10.24)$$

... Trägheitstensor in koord. unabhängiger Form

Komponentendarstellung:

$$\boxed{L_i = \Theta_{ij}\,\omega_j \quad \text{mit} \quad \Theta_{ij} = \underline{e}_i \cdot \underline{\underline{\Theta}}\,\underline{e}_j = \sum_\nu m_\nu \left[|\underline{r}_\nu|^2 \delta_{ij} - x_{\nu i} x_{\nu j} \right]} \quad (10.25)$$

· symmetr. Tensor 2. St.: $\Theta_{ij} = \Theta_{ji} \longrightarrow$ 6 unabh. Elemente

Hauptdiagonale: $\Theta_{11}, \Theta_{22}, \Theta_{33}$ ... Trägheitsmomente

Nebendiagonale: $\Theta_{12}, \Theta_{23}, \Theta_{13}$ .. Deviationsmomente

$\longrightarrow$ Lagerkräfte Bsp. Rad

· Trägheitsmoment bzgl. Achse $\underline{\hat{\nu}}$ mit $|\underline{\hat{\nu}}| = 1$:

$$\Theta_{\nu\nu} = \underline{\hat{\nu}} \cdot \underline{\underline{\Theta}}\,\underline{\hat{\nu}} \quad (10.26)$$

· $\underline{\underline{\Theta}} = \underline{\underline{\Theta}}(t)$. Wo steckt Zeitabhängigkeit relativ zu IS?

(i) ONB des KS: $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ ... körperfest!

$$\underline{\underline{\Theta}}(t) = \Theta_{ij}\, \underline{e}_i(t) \otimes \underline{e}_j(t), \quad \Theta_{ij} = \underline{e}_i(t) \cdot \underline{\underline{\Theta}}\, \underline{e}_j(t) \quad (10.27)$$

... zeitunabhängig

(ii) ONB des IS: $\{\underline{e}_{I1}, \underline{e}_{I2}, \underline{e}_{I3}\}$ .. raumfest

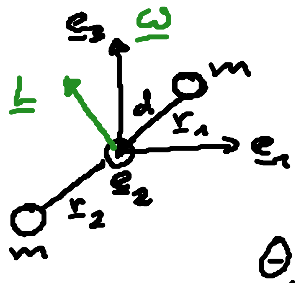
$$\underline{\underline{\Theta}}(t) = \Theta_{Iij}(t)\, \underline{e}_{Ii} \otimes \underline{e}_{Ij}, \quad \Theta_{Iij}(t) = \underline{e}_{Ii} \cdot \underline{\underline{\Theta}}(t)\, \underline{e}_{Ij} \quad (10.28)$$

... zeitabh.

- kontinuierliche Massenverteilung:

$$\underline{\underline{\Theta}} \underset{(10.9)}{=} \int \underbrace{d^3r\, \rho(\underline{r})}_{dm} \left[|\underline{r}|^2 \underline{\underline{1}} - \underline{r} \otimes \underline{r} \right] \quad (10.29)$$

- Bsp.: „Hantel"



$$\underline{r}_1 = \frac{d}{\sqrt{2}} (\underline{e}_1 + \underline{e}_3) \quad , \; |\underline{r}_i| = d$$

$$\underline{r}_2 = -\frac{d}{\sqrt{2}} (\underline{e}_1 + \underline{e}_3)$$

$$\Theta_{11} = \Theta_{33} = m d^2 \, 2\left(1 - \tfrac{1}{2}\right) = m d^2$$

$$\Theta_{13} = \Theta_{31} = m d^2 \, 2\left(-\tfrac{1}{2}\right) = -m d^2$$

$$\Theta_{22} = m d^2 \, 2 \cdot 1 = 2 m d^2$$

$$\Theta_{ij} = 0 \, , \text{ sonst}$$

$$\longrightarrow \underline{\underline{\Theta}} = m d^2 \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad (10.30)$$

$$\underline{\omega} = \omega\, \underline{e}_3 : \quad \underline{L} = \underline{\underline{\Theta}} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = m d^2 \omega \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{L} \not\parallel \underline{\omega} \longrightarrow \frac{d\underline{L}}{dt} \neq 0 \longleftrightarrow \text{Lagerkräften}$$

e) Kinetische Energie: $T = \frac{1}{2} \sum_\nu m_\nu \dot{\underline{r}}_{I\nu}^2$

mit $\dot{\underline{r}}_{I\nu} = \dot{\underline{R}} + \underline{\omega} \times \underline{r}_\nu$

(i) $\underline{R} = \underline{R}_s : \quad \xrightarrow{\quad} \sum_\nu m_\nu \underline{r}_\nu = 0$

$$\boxed{\begin{aligned} T &= T_{trans} + T_{rot} \\ &= \frac{1}{2} M \dot{R}_s^2 + \frac{1}{2} \sum_\nu m_\nu (\underline{\omega} \times \underline{r}_\nu)^2 \end{aligned}} \quad (10.31)$$

Untersuche: $\underline{\omega} \cdot \underline{L} \underset{(10.28)}{=} \sum_\nu m_\nu \underbrace{\underline{\omega} \cdot [\underline{r}_\nu \times (\underline{\omega} \times \underline{r}_\nu)]}_{\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{c} \cdot (\underline{a} \times \underline{b})}$

$$= \sum_\nu m_\nu (\underline{\omega} \times \underline{r}_\nu)^2$$

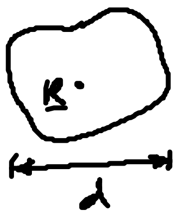
$$\longrightarrow \boxed{T_{rot} = \frac{1}{2} \underline{\omega} \cdot \underline{L} = \frac{1}{2} \underline{\omega} \cdot \underline{\underline{\Theta}}\, \underline{\omega} = \frac{1}{2} \omega_i \Theta_{ij} \omega_j} \quad (10.32)$$

... für Rotation um Aufpkt.

$$\left[\text{vgl. } T_{\text{trans}} = \frac{1}{2} m v^2\right]$$

(ii) $\dot{\underline{R}} = 0$: $\boxed{T = T_{\text{rot}}}$ (10.33)

f) <u>potentielle Energie:</u> $U(\underline{r}_{I\nu})$ für m_ν, $\nu = 1, \ldots, N$



$$\underline{r}_{I\nu} = \underline{R} + \underline{r}_\nu \quad \text{mit } |\underline{r}_\nu| \leq d$$

$$U(\underline{r}_{I\nu}) \overset{\text{Taylor}}{=} U(\underline{R}) + \underline{r}_\nu \cdot \operatorname{grad}_\nu U(\underline{r}_{I\nu})\Big|_{\underline{R}}$$

falls: $d\,\left|\operatorname{grad}_\nu U(\underline{r}_{I\nu})\big|_{\underline{R}}\right| \ll U(\underline{R})$

$$\longrightarrow \boxed{U(\underline{r}_{I\nu}) \approx U(\underline{R})} \qquad (10.34)$$

g) <u>Eigenschaften von $\underline{\underline{\Theta}}$</u>:

(1) <u>Hauptachsentrafo</u>.

$$\boxed{\begin{array}{l} \underline{\underline{\Theta}}\, \underline{e}^{(i)} = \Theta_i\, \underline{e}^{(i)} \\ \left.\begin{array}{l} \Theta_i \geq 0 \ \ldots \text{ Hauptträgheitsmomente} \\ \{\underline{e}^{(1)}, \underline{e}^{(2)}, \underline{e}^{(3)}\} \ldots \text{ Hauptachsensystem} \end{array}\right\} \text{von } \underline{\underline{\Theta}} \end{array}} \qquad (10.35)$$

NB: kinet. Energie $\geq 0 \longrightarrow \Theta_i \geq 0$

mit $\boxed{\underline{e}^{(i)} \cdot \underline{e}^{(j)} = \delta_{ij}}$

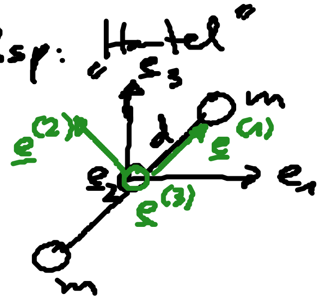
- Diagonalgestalt von $\underline{\underline{\Theta}}$:

$$\tilde{\Theta}_{ij} = \underline{e}^{(i)} \cdot \underbrace{\underline{\underline{\Theta}}\, \underline{e}^{(j)}}_{\Theta_j \underline{e}^{(j)}} = \underbrace{\Theta_j\, \delta_{ij}}_{\text{keine Summe!}} \longrightarrow \tilde{\underline{\underline{\Theta}}} = \begin{pmatrix} \Theta_1 & & 0 \\ & \Theta_2 & \\ 0 & & \Theta_3 \end{pmatrix} \qquad (10.36)$$

$$\text{also: } \underline{\underline{\Theta}} = \sum_{i=1}^{3} \underset{\text{3 EW}}{\Theta_i}\, \underbrace{\underline{e}^{(i)} \otimes \underline{e}^{(i)}}_{\text{3 Eulerwinkel}} \qquad (10.37)$$

3 Eulerwinkel $\longrightarrow$ 6 unabh. Komp von $\underline{\underline{\Theta}}$

- Fälle: (i) $\Theta_1 \neq \Theta_2 \neq \Theta_3$: allgemeiner unsymmetr. Kreisel
  (ii) $\Theta_1 = \Theta_2 \neq \Theta_3$: achsen " " („Rotationsellipsoid")
  (iii) $\Theta_1 = \Theta_2 = \Theta_3$: Würfel oder Kugel
- physikal.: $\underline{\omega} \parallel \underline{e}^{(i)} \longrightarrow \underline{L} \parallel \underline{\omega}$ ... stabile Drehrichtungen!
- Bsp.: „Hantel"



$$\underline{\underline{\Theta}} = md^2 \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

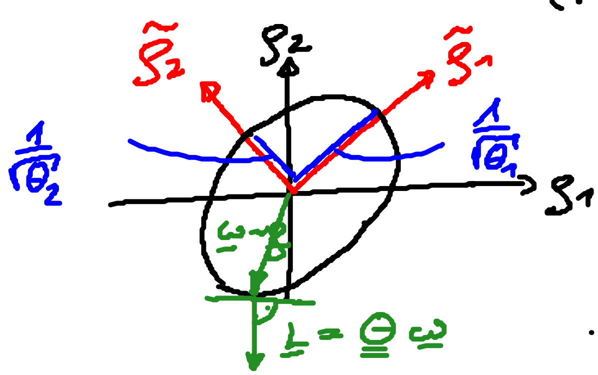
$$\left.\begin{aligned} \underline{e}^{(1)} &= \tfrac{1}{\sqrt{2}}(\underline{e}_1 + \underline{e}_3) = \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \Theta_1 = 0 \\ \underline{e}^{(2)} &= \tfrac{1}{\sqrt{2}}(-\underline{e}_1 + \underline{e}_3) = \tfrac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}, \quad \Theta_2 = 2md^2 \\ \underline{e}^{(3)} &= \underline{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \Theta_3 = 2md^2 \end{aligned}\right\} \quad \tilde{\underline{\underline{\Theta}}} = 2md^2 \begin{pmatrix}0 & & \\ & 1 & \\ & & 1\end{pmatrix}$$

(2) <u>Trägheitsellipsoid</u> = Fläche konst. Rot. energie

- $T = \frac{1}{2}\,\underline{\omega} \cdot \underline{\underline{\Theta}}\,\underline{\omega}$ mit $\underline{\rho} = \dfrac{\underline{\omega}}{\sqrt{2T}}$ (10.38)

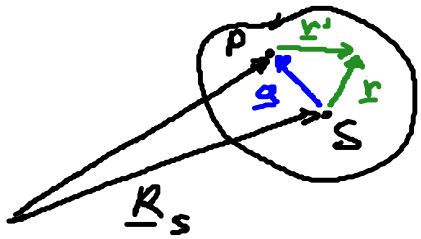
$$\longrightarrow \boxed{1 = \rho_i \Theta_{ij} \rho_j \xrightarrow[\text{trafo}]{\text{Hauptachsen-}} 1 = \Theta_1 \tilde{\rho}_1^2 + \Theta_2 \tilde{\rho}_2^2 + \Theta_3 \tilde{\rho}_3^2}$$

... Trägheitsellipsoid mit Halbachsen $\frac{1}{\sqrt{\Theta_i}}$
(Fläche 2. Grades)



... <u>Poinsot'sche Konstruktion</u>
(Beweis: Übungen)

(3) Satz von Steiner:



$\underline{\underline{\Theta}}^{(S)}$ bezogen auf S bekannt
Schwerpkt

$$\longrightarrow \quad \boxed{\underline{\underline{\Theta}}^{(P)} = M\left(|\underline{a}|^2 \underline{\underline{1}} - \underline{a} \otimes \underline{a}\right) + \underline{\underline{\Theta}}^{(S)}} \qquad (10.39)$$

Gesamtmasse ... Trägheitstensor bzgl. Punkt P

Beweis: Übungen

10.3 Dynamik des starren Körpers (II): Eulersche Gleichungen

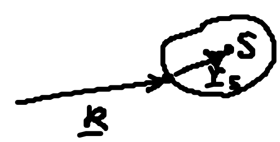
a) Dynam. Grundgleichungen

- Impulssatz:

(10.13)
(8.8)
$$\boxed{\dot{\underline{P}} = \sum_{\nu} \underline{F}_{\nu}^{(a)} = \underline{F}^{(a)}}$$

(i) $\underline{R} = \underline{R}_S$: $\quad \underline{P} = M \dot{\underline{R}}_S \qquad (10.16)$

(ii) $\dot{\underline{R}} = 0$: $\quad \underline{P} = \underline{\omega} \times M \underline{r}_S \qquad (10.17)$



- Drehimpulssatz:

$$\underset{(8.14)}{\overset{(10.14)}{}}\quad \dot{\underline{L}}_I = \underline{D}_I^{(a)} = \sum_\nu \underline{r}_{I\nu} \times \underline{F}_\nu^{(a)} \underset{\underline{r}_{I\nu} = \underline{R} + \underline{r}_\nu}{=\!=} \underline{R} \times \underline{F}^{(a)} + \underline{D} \qquad (10.40)$$

mit $\boxed{\underline{D} = \sum_\nu \underline{r}_\nu \times \underline{F}_\nu^{(a)}}$ (10.41)

... Drehmoment bzgl. Aufpkt. $\underline{R}$

$$\overset{(10.18)}{\underset{(10.21)}{=}} \dot{\underline{L}}_s + \dot{\underline{L}}$$

(i) $\underline{R} = \underline{R}_s$: $\quad \dot{\underline{L}}_s = (\underline{R}_s \times \underbrace{M \dot{\underline{R}}_s}_{\underline{P}})^{\cdot} = \underline{R}_s \times \dot{\underline{P}} \overset{}{\underset{(10.13)}{=}} \underline{R}_s \times \underline{F}^{(a)}$ (10.42)

$\dot{\underline{R}}_s \times \dot{\underline{R}}_s = 0$

(ii) $\dot{\underline{R}} = 0$: $\quad \dot{\underline{L}}_s = \underline{R} \times \dot{\underline{P}} = \underline{R} \times \underline{F}^{(a)}$ (10.43)

$$\Longrightarrow \quad \boxed{\dot{\underline{L}} = \underline{D}}$$

... Drehimpulssatz für starren Körper bzgl. Aufpkt. $\underline{R}$ $\longrightarrow$ Rotationsbewegung

[vgl. Newton: $\dot{\underline{p}} = \underline{F}$ mit $\underline{p} = m \underline{v} \longleftrightarrow \underline{L} = \underline{\Theta}\, \underline{\omega}$

$\underline{F} \longleftrightarrow \underline{D}$]