14.5 Kanonische Transformationen

[lat. bestangepaßt]

• Motivierende Idee:

alle q_j zyklisch & $\frac{\partial H}{\partial t} = 0$

$$\longrightarrow \dot{p}_j = -\frac{\partial H}{\partial q_j} = 0 \longrightarrow p_j = \alpha_j = \text{konst.}$$

$$\longrightarrow H = H(\alpha_1, \dots, \alpha_f) \longrightarrow \dot{q}_j = \frac{\partial H}{\partial \alpha_j} = \omega_j(\alpha_1 \dots \alpha_f) = \text{konst.}$$

$$\longrightarrow \boxed{q_j = \omega_j t + \beta_j} \quad (14.41)$$

Bem.: (i) (14.41) definiert integrable Systeme (..., die Ausnahme" ...)

(ii) die Regel: nicht-integrable Systeme Bsp. 3-Körper-Problem

Ziel: Trafo: $\{q_j, p_j\} \longrightarrow \{Q_k, P_k\}$, so daß alle Q_k zyklisch!

• Bisher: $\boxed{\text{Punkttransformationen: } Q_k = Q_k(\{q_j\}, t)}$

Beh: $\boxed{\text{Punkttrafos lassen die Lagrange Gln. forminvariant}}$

Bew.: Umkehrung: $q_j = q_j(\{Q_k\}, t) \longrightarrow \dot{q}_j = \sum_k \frac{\partial q_j}{\partial Q_k} \dot{Q}_k + \frac{\partial q_j}{\partial t} \quad (*)$

Zeige: $\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_k} - \frac{\partial L}{\partial Q_k} = 0$!

$$= \underbrace{\sum_j \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_j} \underbrace{\frac{\partial \dot{q}_j}{\partial \dot{Q}_k}}_{(*)\ \frac{\partial q_j}{\partial Q_k}}\Big)}_{\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\right)\frac{\partial q_j}{\partial Q_k} + \cancel{\frac{\partial L}{\partial \dot{q}_j}\frac{d}{dt}\frac{\partial q_j}{\partial Q_k}}} - \sum_j \Big(\frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial Q_k} + \cancel{\frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial Q_k}}\Big)$$

$$= \sum_j \underbrace{\Big(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j}\Big)}_{=0}\frac{\partial q_j}{\partial Q_k} = 0 \checkmark$$

• jetzt

$$\boxed{\begin{array}{l} \underline{\text{kanonische Trafos:}} \quad Q_k = Q_k(\{q_j\}, \{p_j\}, t) \\ \qquad\qquad\qquad\qquad\quad P_k = P_k(\{q_j\}, \{p_j\}, t) \\ \text{wenn Hamiltonsche Bew.gln. forminvariant bleiben} \\ \text{bzgl. neuer Hamiltonfunktion } \bar{H} = \bar{H}(\{Q_k\}, \{P_k\}, t) \end{array}} \qquad (14.43)$$

äquivalent: $L(\{q_j\}, \{\dot{q}_j\}, t) \underset{\text{Trafo}}{\overset{\text{kanon.}}{=}} L(\{Q_k\}, \{\dot{Q}_k\}, t) + \underbrace{\frac{d}{dt} F}_{\text{vgl. Kap. 13.5 c)}}$

$$\xleftrightarrow[\text{Trafo}]{\text{Legendre}} \quad \boxed{\sum_j p_j \dot{q}_j - H = \sum_j P_j \dot{Q}_j - \bar{H} + \frac{d}{dt} F} \qquad (14.45)$$

• $\underline{\text{erzeugende Funktion } F}$:

(i) bestimmt kanonische Trafo!

(ii) soll also von $\underbrace{\{q_j\}, \{p_j\}, \{Q_j\}, \{P_j\}}$ abhängen

2f unabhängig

2f abhängige Variable [vgl. (14.43)]

$\longrightarrow$ 4 Möglichkeiten

$$\begin{array}{ll} F_1(\{q_j\}, \{Q_j\}, t) & F_2(\{q_j\}, \{P_j\}, t) \\ F_3(\{p_j\}, \{Q_j\}, t) & F_4(\{p_j\}, \{P_j\}, t) \end{array} \qquad (14.46)$$

(i) $F_1(\{q_j\},\{Q_j\},t)$:

Bilde: $\frac{d}{dt}F_1 = \sum_j\left(\frac{\partial F_1}{\partial q_j}\dot{q}_j + \frac{\partial F_1}{\partial Q_j}\dot{Q}_j\right) + \frac{\partial F_1}{\partial t}$

in (14.45) & Koeff.-vgl. bei $\dot{q}_j$, $\dot{Q}_j$, da $\{q_j\}$, $\{Q_j\}$ unabh. Variable

$$
\begin{array}{l}
\xrightarrow{\text{für } \dot{q}_j} \\[2em]
\xrightarrow{\text{für } \dot{Q}_j} \\[2em]
\xrightarrow{\text{Rest}}
\end{array}
\boxed{\begin{aligned}
p_j &= \frac{\partial F_1}{\partial q_j} \\
P_j &= -\frac{\partial F_1}{\partial Q_j} \\
\bar{H} &= H + \frac{\partial F_1}{\partial t}
\end{aligned}}
\begin{array}{l}
\xrightarrow{\text{Umkehrung}} Q_k = Q_k(\{q_j\},\{p_j\},t) \\[1em]
\longrightarrow P_k = P_k(\{q_j\},\{Q_j\},t) \\
\qquad\quad = P_k(\{q_j\},\{p_j\},t)
\end{array}
$$

… F_1 erzeugt kanon. Trafo

(ii) $F_2(\{q_j\},\{P_j\},t) \underset{\text{Trafo}}{\overset{\text{Leg.}}{=}} F_1 + \sum_j P_j Q_j$ (14.48)

$Q_j \to -P_j$

$P_j = -\frac{\partial F_1}{\partial Q_j}$

$$
\xrightarrow{\text{Theorie der}} \text{Legendre Trafo}
\quad
\boxed{\begin{aligned}
p_j &= \frac{\partial F_2}{\partial q_j} \\
Q_j &= +\frac{\partial F_2}{\partial P_j} \\
\bar{H} &= H + \frac{\partial F_2}{\partial t}
\end{aligned}}
$$

… F_2 erzeugt kanon. Trafo.

(iii) $F_3(\{p_j\},\{Q_j\},t) \underset{\text{Trafo}}{\overset{\text{Leg}}{=}} F_1 - \sum_j p_j q_j$ (14.50)

$$\longrightarrow \boxed{\begin{aligned} q_j &= -\frac{\partial F_3}{\partial p_j} \\ P_j &= -\frac{\partial F_3}{\partial Q_j} \\ \bar{H} &= H + \frac{\partial F_3}{\partial t} \end{aligned}} \qquad (14.51)$$

... F_3 erzeugt kanon. Trafo

(iv) $F_4(\{p_j\},\{P_j\},t) \underset{\text{Trafo}}{\overset{\text{Leg.}}{=}} F_1 + \sum_j P_j Q_j - \sum_j p_j q_j \qquad (14.52)$

$$\longrightarrow \boxed{\begin{aligned} q_i &= -\frac{\partial F_4}{\partial p_j} \\ Q_j &= \frac{\partial F_4}{\partial P_j} \\ \bar{H} &= H + \frac{\partial F_4}{\partial t} \end{aligned}} \qquad (14.53)$$

... F_4 erzeugt kanon. Trafo

• Beispiele:

(i) identische Trafo: $F_2 = \sum_j q_j P_j \longrightarrow$

$$p_j = \frac{\partial F_2}{\partial q_j} = P_j$$

$$Q_j = \frac{\partial F_2}{\partial P_j} = q_j \qquad !$$

$$\bar{H} = H$$

(ii) $\boxed{\text{Punkttrafo} = \text{kanon. Trafo}}$!

$$F_2 = \sum_j f_j(\{q_k\},t) P_j \xrightarrow[Q_j = \frac{\partial F_2}{\partial P_j}]{} Q_j = f_j(\{q_k\},t), \quad \bar{H} = H + \frac{\partial F_2}{\partial t}$$

(iii) $F_1 = \sum_j q_j Q_j \longrightarrow$

$$\left.\begin{aligned} p_j &= \frac{\partial F_1}{\partial q_j} = Q_j \\ P_j &= -\frac{\partial F_1}{\partial Q_j} = -q_j \end{aligned}\right\} \text{Vertauschung von Impulsen}$$

(iv) harmonischer Oszillator: $(x, p) = (q, p)$

$$\boxed{H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2} \quad (14.54)$$ mit Federkonst. $f = m\omega^2$

$$F_1 = \frac{m}{2}\omega q^2 \cot Q \longrightarrow p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q \quad (1)$$

$$P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2} \frac{1}{\sin^2 Q} \quad (2)$$

$$\left.\begin{aligned} (2) &\longrightarrow q = \sqrt{\frac{2P}{m\omega}} \sin Q \quad (3) \\ (3) \text{ in } (1) &\longrightarrow p = \sqrt{2m\omega P} \cos Q \quad (4) \end{aligned}\right\} \text{kanon. Umkehrtrafo !!}$$

(3),(4) in H: $\quad H = \bar{H} = \omega P \cos^2 Q + \omega P \sin^2 Q$

$$\longrightarrow \boxed{H = \omega P \longleftrightarrow \text{Impuls } P = \frac{\text{Energie } E = H}{\omega}} \quad !$$

Q ... zyklische Koord.: $\longrightarrow$ Bewgl. $\dot{Q} = \frac{\partial H}{\partial P} = \omega$

$$\begin{aligned} &\longrightarrow \boxed{Q = \omega t + \alpha} \\ &\longrightarrow \boxed{q \overset{(3)}{=} \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \alpha)} \quad \checkmark \end{aligned}$$

15. Hamilton-Jacobische Theorie

- nur Skizzierung!
- Grundidee:

Ges: kanonische Trafo auf
$$\boxed{\begin{aligned} Q_k &= \beta_k = \text{konst.} \overset{\text{z.B.}}{=} q_k(0) \\ P_k &= \underbrace{\alpha_k}_{\text{2f Integrations-konst. des Systems}} = \text{konst.} = p_k(0) \end{aligned}} \quad (15.1)$$

damit Hamilton Bewgl.

$$\left.\begin{aligned} 0 &= \dot{Q}_k = \frac{\partial \bar{H}}{\partial P_k} \\ 0 &= \dot{P}_k = -\frac{\partial \bar{H}}{\partial Q_k} \\ &\text{mit } \bar{H} = H + \frac{\partial F}{\partial t} \end{aligned}\right\} \quad (14.4)$$

o.B.d.A. $$\boxed{\bar{H} = H + \frac{\partial F}{\partial t} = 0} \quad (15.2)$$

- erzeugende Fkt.? [vgl. Kap. 14.5]

$$\boxed{F_2(\{q_j\}, \{P_j\} = \{\alpha_j\}, t) = S(\{q_j\}, \{\alpha_j\}, t)} \quad (15.3)$$

... Hamilton-Jacobische-Wirkungsfkt.
(Hamiltonsche Prinzipalfkt.)

mit $$\boxed{p_j = \frac{\partial S}{\partial q_j}} \quad (14.49)$$

in (15.2) ⟶ $$\boxed{H\left(\{q_j\}, \left\{\frac{\partial S}{\partial q_j}\right\}, t\right) + \frac{\partial S}{\partial t} = 0} \quad (15.4)$$

... Hamilton-Jacobische-Dgl. für S!

<u>Bem</u>: Hamiltonsche Theorie
Löse 2f gewöhnl. Dgl.
1. Ordnung in Zeit

$$\dot{q}_j = \frac{\partial H}{\partial p_j}$$
$$\dot{p}_j = -\frac{\partial H}{\partial q_j}$$

$\underset{\text{zu}}{\overset{\text{äquivalent}}{\longleftrightarrow}}$

Hamilton-Jacobische Theorie
S als Lsg. von (15.4)
nichtlineare partielle Dgl. 1. Ordnung
in f+1 Variable q_j, t