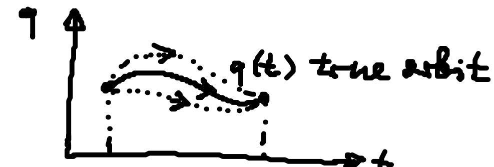
English Summary:

2.2 Hamilton's Principle

2.2.1 Variational Principles



$\delta t = 0$

$\delta q\big|_{t_1} = \delta q\big|_{t_2} = 0$

extremum of functional

$$\delta I[q] = \delta \int_{t_1}^{t_2} dt\, F(q(t), \dot{q}(t), t) \overset{!}{=} 0$$

$$\Leftrightarrow \quad \frac{d}{dt}\frac{\partial F}{\partial \dot{q}} - \frac{\partial F}{\partial q} = 0 \qquad \text{Euler-Lagrange eqs.}$$

2.2.2 Hamilton's Action Principle (holonomic constraints, conservative forces)

action $W := \int_{t_1}^{t_2} dt\, L(q_1(t)\ldots, \dot{q}_1(t)\ldots, t)$

$$\boxed{\delta W = 0} \quad \Leftrightarrow \quad \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0} \qquad k = 1, \ldots, f$$

Principle of least action — Lagrange eqs. of the 2nd kind

2.2.3 Eichtransformationen der Lagrangefunktion

Die Lagrangefkt. ist durch die Lagrangegln. nicht eindeutig festgelegt.

Betrachte <u>geladenes Teilchen im elektromagn. Feld</u>

$\underline{q} = (q_1, q_2, q_3) = (x_1, x_2, x_3) \qquad e \gtrless 0$ Ladung

Bewegungsgl. $m\ddot{\underline{q}} = e\underline{E}(\underline{q},t) + e\dot{\underline{q}} \times \underline{B}(\underline{q},t) \qquad \circledast$

<u>Lorentzkraft</u> (nichtkonservativ!)

el. Feld $\underline{E}(\underline{q},t) = -\underline{\nabla}\phi(\underline{q},t) - \frac{\partial}{\partial t}\underline{A}(\underline{q},t)$

magn. Indukt. $\underline{B}(\underline{q},t) = \underline{\nabla} \times \underline{A}(\underline{q},t)$

skalares Pot. ϕ, Vektorpot. $\underline{A}$

<u>Ziel</u>: Suche $L(\underline{q},\dot{\underline{q}},t)$ so, dass die Lagrangegln. $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0$ $\circledast$ ergeben

Ansatz: $\boxed{L(\underline{q},\dot{\underline{q}},t) = \frac{m}{2}\dot{\underline{q}}^2 + e\left(\dot{\underline{q}}\cdot\underline{A}(\underline{q},t) - \phi(\underline{q},t)\right)}$

Probe: $\frac{\partial L}{\partial \dot{q}_k} = m\dot{q}_k + eA_k$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = m\ddot{q}_k + e\frac{d}{dt}A_k(\underline{q}(t),t)$ totale Zeitableitung längs Bahn $\underline{q}(t)$

$= m\ddot{q}_k + e\left(\frac{\partial}{\partial t}A_k + \sum_l \frac{\partial A_k}{\partial q_l}\dot{q}_l\right)$

$= m\ddot{q}_k + e\left(\frac{\partial}{\partial t}A_k + \dot{\underline{q}}\cdot\underline{\nabla}A_k\right)$

$\frac{\partial L}{\partial q_k} = e\left[\frac{\partial}{\partial q_k}(\dot{\underline{q}}\cdot\underline{A}) - \frac{\partial}{\partial q_k}\phi\right]$

$$\Rightarrow 0 = m\ddot{q}_k + \underbrace{e\left(\frac{\partial}{\partial t}A_k + \frac{\partial}{\partial q_k}\phi\right)}_{-E_k} + e\underbrace{\left[(\dot{\underline{q}}\cdot\underline{\nabla})A_k - \frac{\partial}{\partial q_k}(\dot{\underline{q}}\cdot\underline{A})\right]}_{-[\dot{\underline{q}}\times(\underline{\nabla}\times\underline{A})]_k = -(\dot{\underline{q}}\times\underline{B})_k}$$

$\underline{a}\times(\underline{b}\times\underline{c}) = \underline{b}(\underline{a}\underline{c}) - \underline{c}(\underline{a}\underline{b})$

$-\underline{a}\times(\underline{b}\times\underline{c}) = (\underline{a}\underline{b})\underline{c} - \underline{b}(\underline{a}\underline{c})$

$\square$

Umeichung der Potenziale mit bel. Eichfkt. χ:

$$\underline{A}(\underline{q},t) \longrightarrow \underline{A}'(\underline{q},t) = \underline{A}(\underline{q},t) + \underline{\nabla}\chi(\underline{q},t)$$

$$\phi(\underline{q},t) \longrightarrow \phi'(\underline{q},t) = \phi(\underline{q},t) - \frac{\partial}{\partial t}\chi(\underline{q},t)$$

Dabei ändern sich die Felder nicht:

$$\underline{E}' = -\underline{\nabla}\phi' - \frac{\partial}{\partial t}\underline{A}' = -\underline{\nabla}\phi + \cancel{\underline{\nabla}\frac{\partial}{\partial t}\chi} - \frac{\partial}{\partial t}\underline{A} - \cancel{\frac{\partial}{\partial t}\underline{\nabla}\chi} = \underline{E}$$

$$\underline{B}' = \underline{\nabla}\times\underline{A}' = \underline{\nabla}\times\underline{A} + \underbrace{\underline{\nabla}\times\underline{\nabla}\chi}_{0} = \underline{B}$$

Lagrangefkt.:

$$L' = \frac{m}{2}\dot{\underline{q}}^2 + e(\dot{\underline{q}}\cdot\underline{A}' - \phi') = \frac{m}{2}\dot{q}^2 + e(\dot{\underline{q}}\cdot\underline{A} + \dot{\underline{q}}\cdot\underline{\nabla}\chi - \phi + \dot{\chi})$$

$$= L + e\left(\frac{\partial\chi}{\partial t} + \dot{\underline{q}}\cdot\underline{\nabla}\chi\right)$$

$$= L + \frac{d}{dt}(e\chi(\underline{q},t))$$

L' führt zu denselben Lagrangegln. wie L!

Die Eichtransformationen $\boxed{L \to L' = L + \frac{d}{dt}M(\underline{q},t)}$

mit bel. Eichfkt. M lässt die Lagrangegln. invariant.

Allgemein gilt: Sei $M(q_1,\dots,q_f,t) \in C^3$ beliebig

und $\boxed{L'(q_1\dots,\dot{q}_1\dots,t) = L(q_1\dots,\dot{q}_1\dots,t) + \sum_{k=1}^{f}\frac{\partial M}{\partial q_k}\dot{q}_k + \frac{\partial M}{\partial t}}$

Dann erfüllen $\{q_k(t)\}$ das Hamilton'sche Prinzip

$$\boxed{\delta\int L'\,dt = 0} \Leftrightarrow \boxed{\delta\int L\,dt = 0}$$

d.h. die Euler-Lagrange-Dgl. sind invariant

Beweis:

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot q_k} - \frac{\partial L'}{\partial q_k} = \frac{d}{dt}\frac{\partial L}{\partial \dot q_k} - \frac{\partial L}{\partial q_k} + \underbrace{\frac{d}{dt}\underbrace{\frac{\partial}{\partial \dot q_k}\left(\sum_\ell \frac{\partial M}{\partial q_\ell}\dot q_\ell + \frac{\partial M}{\partial t}\right)}_{\frac{\partial M}{\partial q_k}} - \frac{\partial}{\partial q_k}\underbrace{\left(\sum_\ell \frac{\partial M}{\partial q_\ell}\dot q_\ell + \frac{\partial M}{\partial t}\right)}_{\frac{dM}{dt}}}_{0} \qquad \square$$

NB: $M(q_1 \dots q_f, t)$ darf nicht explizit von $\dot q_k$ abhängen!

Beispiel: 1-dim. harmon. Osz.

$$L = \frac{m}{2}\dot q^2 - \frac{m\omega^2}{2}q^2$$

z.B. Eichfkt. $M(q) := \frac{m\omega^2}{2}q^2 \Rightarrow \frac{dM}{dt} = m\omega^2 q\dot q$

$$L' = \frac{m}{2}\dot q^2 - \frac{m\omega^2}{2}\left(q^2 - 2q\dot q\right)$$

Lagrangegln.:

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial L'}{\partial \dot q} &= m\ddot q + m\omega^2\dot q \\ \frac{\partial L'}{\partial q} &= -m\omega^2 q + m\omega^2\dot q \end{aligned}\right\} \ddot q + \omega^2 q = 0$$

2.2.4 Forminvarianz der Lagrangegln.

Betrachte nun eine schwächere Invarianz der Lagrangegln.:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_k} - \frac{\partial L}{\partial q_k} = 0 \quad \rightarrow \quad \frac{d}{dt}\frac{\partial \bar L}{\partial \dot Q_k} - \frac{\partial \bar L}{\partial Q_k} = 0 \qquad \text{Forminvarianz}$$

Frage: Für welche Trafo der generalisierten Koord. $F: \{q_k\} \rightarrow \{Q_k\}$ sind die Lagrangegln. forminvariant?

Satz: Sei $F: \{q_k\} \rightarrow \{Q_i\}$ ein C^2-Diffeomorphismus

d.h. umkehr eindeutige Abbildung, F und F^{-1} 2x stetig diff.bar

$$\left(\begin{aligned} Q_i &= F_i(q_1 \dots q_f, t) \\ q_k &= f_k(Q_1 \dots Q_f, t) \end{aligned} \quad \text{mit } \det\left(\frac{\partial f_k}{\partial Q_i}\right) \neq 0\right)$$

Dann ist $\{Q_k(t)\}$ Lösung der Lagrangegln. zur transformierten Lagrangefkt. $\bar{L}(Q_1, \dots, \dot{Q}_1, \dots, t)$

$$\bar{L}(Q_k, \dot{Q}_k, t) := L(\underbrace{f_k(Q_i, t)}_{q_k}, \underbrace{\sum_i \frac{\partial f_k}{\partial Q_i}\dot{Q}_i + \frac{\partial f_k}{\partial t}}_{\dot{q}_k}, t)$$

$\Longleftrightarrow$ $\{q_k(t)\}$ Lösung der Lagrangegln. zu $L(q_k, \dot{q}_k, t)$

Beweis:

$$\dot{q}_\ell = \sum_i \frac{\partial f_\ell}{\partial Q_i}\dot{Q}_i + \frac{\partial f_\ell}{\partial t}$$

$$\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{Q}_k} = \sum_{\ell=1}^{f} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\ell}\frac{\partial \dot{q}_\ell}{\partial \dot{Q}_k}\right) = \sum_\ell \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\ell}\frac{\partial q_\ell}{\partial Q_k}\right)$$

$$= \sum_\ell \left\{\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\ell}\right)\right]\frac{\partial q_\ell}{\partial Q_k} + \frac{\partial L}{\partial \dot{q}_\ell}\frac{d}{dt}\left(\frac{\partial q_\ell}{\partial Q_k}\right)\right\}$$

$$= \sum_\ell \left\{\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\ell}\right)\right]\frac{\partial q_\ell}{\partial Q_k} + \frac{\partial L}{\partial \dot{q}_\ell}\frac{\partial \dot{q}_\ell}{\partial Q_k}\right\}$$

$$\frac{\partial \bar{L}}{\partial Q_k} = \sum_{\ell=1}^{f}\left[\frac{\partial L}{\partial q_\ell}\frac{\partial q_\ell}{\partial Q_k} + \frac{\partial L}{\partial \dot{q}_\ell}\frac{\partial \dot{q}_\ell}{\partial Q_k}\right]$$

$$\Rightarrow \underbrace{\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{Q}_k} - \frac{\partial \bar{L}}{\partial Q_k}}_{\text{Variationsabl. ist „kovariant" unter diffeomorphen Transformationen der generalisierten Koord.}} = \sum_{\ell=1}^{f} \underbrace{\frac{\partial q_\ell}{\partial Q_k}}_{\text{Transformationsmatrix (nichtsingulär, d.h. det} \neq 0)}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\ell}\right) - \frac{\partial L}{\partial q_\ell}\right]$$

$\square$

$$\boxed{\text{Es gibt unendl. viele äquivalente Sätze generalisierter Koordinaten!}}$$