

English Summary:

2.2 Hamilton's Principle

2.2.1 Variational Principles

extremum of functional

$$\delta I[q] = \delta \int_{t_1}^{t_2} dt F(q(t), \dot{q}(t), t) \stackrel{!}{=} 0$$

$$\Leftrightarrow \frac{d}{dt} \frac{\partial F}{\partial \dot{q}} - \frac{\partial F}{\partial q} = 0 \quad \text{Euler-Lagrange eqs.}$$

2.2.2 Hamilton's Action Principle (holonomic constraints, conservative forces)

$$\text{action } W := \int_{t_1}^{t_2} dt L(q(t), \dot{q}(t), t)$$

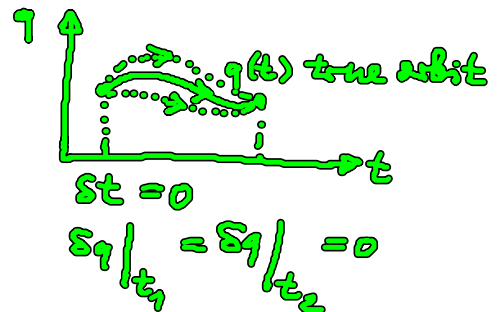
$$\boxed{\delta W = 0}$$

Principle of least action

\Leftrightarrow

$$\boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0}$$

Lagrange eqs. of the 2nd kind



2.2.3 Eichtransformationen der Lagrangefunktion

Die Lagrangefkt. ist durch die Lagrangegl. nicht eindeutig festgelegt.

Betrachte geladenes Teilchen im elektromagn. Feld

$$\underline{q} = (q_1, q_2, q_3) = (x_1, x_2, x_3) \quad e \geq 0 \quad \text{Ladung}$$

$$\text{Bewegungsgl. } m \underline{\ddot{q}} = e \underline{E}(\underline{q}, t) + e \underline{\dot{q}} \times \underline{B}(\underline{q}, t) \quad (*)$$

Lorentzkraft (nichtkonservativ!)

$$\text{el. Feld } \underline{E}(\underline{q}, t) = - \underline{\nabla} \phi(\underline{q}, t) - \frac{\partial}{\partial t} \underline{A}(\underline{q}, t)$$

$$\text{magn. Indukt. } \underline{B}(\underline{q}, t) = \underline{\nabla} \times \underline{A}(\underline{q}, t)$$

skalares Pot. ϕ , Vektorpot. \underline{A}

Ziel: Suche $L(\underline{q}, \underline{\dot{q}}, t)$ so, dass die Lagrangegl. $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0$ ergeben

$$\text{Ansatz: } \boxed{L(\underline{q}, \underline{\dot{q}}, t) = \frac{m}{2} \dot{\underline{q}}^2 + e (\dot{\underline{q}} \cdot \underline{A}(\underline{q}, t) - \phi(\underline{q}, t))}$$

$$\text{Probe: } \frac{\partial L}{\partial \dot{q}_k} = m \dot{q}_k + e A_k$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = m \ddot{q}_k + e \frac{d}{dt} A_k(\underline{q}(t), t) \quad \text{totale Zeitableitung längs Bahn } \underline{q}(t)$$

$$= m \ddot{q}_k + e \left(\frac{\partial}{\partial t} A_k + \sum_i \frac{\partial A_k}{\partial q_i} \dot{q}_i \right)$$

$$= m \ddot{q}_k + e \left(\frac{\partial}{\partial t} A_k + \dot{\underline{q}} \cdot \underline{\nabla} A_k \right)$$

$$\frac{\partial L}{\partial q_k} = e \left[\frac{\partial}{\partial q_k} (\dot{\underline{q}} \cdot \underline{A}) - \frac{\partial}{\partial q_k} \phi \right]$$

$$\Rightarrow 0 = m \ddot{q}_k + e \underbrace{\left(\frac{\partial A_k}{\partial t} + \frac{\partial \phi}{\partial q_k} \right)}_{-E_k} + e \underbrace{\left[(\dot{\underline{q}} \cdot \underline{\nabla}) A_k - \frac{\partial}{\partial q_k} (\dot{\underline{q}} \cdot \underline{A}) \right]}_{-[\dot{\underline{q}} \times (\underline{\nabla} \times \underline{A})]_k = -(\dot{\underline{q}} \times \underline{B})_k}$$

$$\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a} \cdot \underline{c}) - \underline{c}(\underline{a} \cdot \underline{b})$$

$$- \underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{b}) \underline{c} - \underline{b}(\underline{a} \cdot \underline{c})$$

□

Umreichung der Potentiale mit bel. Eichfkt. χ :

$$\underline{A}(q, t) \rightarrow \underline{A}'(q, t) = \underline{A}(q, t) + \underline{\nabla}\chi(q, t)$$

$$\phi(q, t) \rightarrow \phi'(q, t) = \phi(q, t) - \frac{\partial}{\partial t}\chi(q, t)$$

Dabei ändern sich die Felder nicht:

$$\underline{E}' = -\underline{\nabla}\phi' - \frac{\partial}{\partial t}\underline{A}' = -\underline{\nabla}\phi + \underline{\nabla}\frac{\partial}{\partial t}\chi - \frac{\partial}{\partial t}\underline{A} - \frac{\partial}{\partial t}\underline{\nabla}\chi = \underline{E}$$

$$\underline{B}' = \underline{\nabla} \times \underline{A}' = \underline{\nabla} \times \underline{A} + \underbrace{\underline{\nabla} \times \underline{\nabla}\chi}_0 = \underline{B}$$

Lagrangefkt.:

$$L' = \frac{m}{2} \dot{q}^2 + e(\dot{q} \cdot \underline{A}' - \phi') = \frac{m}{2} \dot{q}^2 + e(\dot{q} \cdot \underline{A} + \dot{q} \cdot \underline{\nabla}\chi - \phi + \dot{\chi})$$

$$= L + e \left(\frac{\partial \chi}{\partial t} + \dot{q} \cdot \underline{\nabla}\chi \right)$$

$$= L + \frac{d}{dt} (e\chi(q, t))$$

L' führt zu denselben Lagrangegl. wie L !

Die Eichtransformationen $L \rightarrow L' = L + \frac{d}{dt} M(q, t)$

mit bel. Eichfkt. M lässt die Lagrangegl. invariant.

Allgemein gilt: Sei $M(q_1, \dots, q_f, t) \in C^3$ beliebig

$$\text{und } L'(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t) = L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t) + \sum_{k=1}^f \frac{\partial M}{\partial q_k} \dot{q}_k + \frac{\partial M}{\partial t}$$

Dann erfüllen $\{q_k(t)\}$ das Hamilton'sche Prinzip

$$\delta \int L' dt = 0 \iff \delta \int L dt = 0$$

d.h. die Euler-Lagrange-Dgl. sind invariant

Beweis: $\frac{d}{dt} \frac{\partial L'}{\partial \dot{q}_k} - \frac{\partial L'}{\partial q_k} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} + \underbrace{\frac{d}{dt} \frac{\partial}{\partial \dot{q}_k} \left(\sum_l \frac{\partial M}{\partial \dot{q}_l} \dot{q}_l + \frac{\partial M}{\partial t} \right)}_{\frac{\partial M}{\partial \dot{q}_k}} - \underbrace{\frac{\partial}{\partial q_k} \left(\sum_l \frac{\partial M}{\partial \dot{q}_l} \dot{q}_l + \frac{\partial M}{\partial t} \right)}_{\frac{dM}{dt}}$ \square

NB: $M(q_1 \dots q_f, t)$ darf nicht explizit von \dot{q} abhängen!

Beispiel: 1-dim. harmon. Osz.

$$L = \frac{m}{2} \dot{q}^2 - \frac{m\omega^2}{2} q^2$$

z.B. Einfeld. $M(q) := \frac{m\omega^2}{2} q^2 \Rightarrow \frac{dM}{dt} = m\omega^2 q \dot{q}$

$$L' = \frac{m}{2} \dot{q}^2 - \frac{m\omega^2}{2} (q^2 - 2q\dot{q})$$

Lagrange gl.: $\left. \begin{aligned} \frac{d}{dt} \frac{\partial L'}{\partial \dot{q}} &= m\dot{q} + m\omega^2 q \\ \frac{\partial L'}{\partial q} &= -m\omega^2 q + m\omega^2 \dot{q} \end{aligned} \right\} \ddot{q} + \omega^2 q = 0$

2.2.4 Forminvarianz der Lagrange gl.

Betrachte nun eine schwächere Invarianz der Lagrange gl.:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0 \rightarrow \frac{d}{dt} \frac{\partial \bar{L}}{\partial \dot{Q}_k} - \frac{\partial \bar{L}}{\partial Q_k} = 0 \quad \underline{\text{Forminvarianz}}$$

Frage: Für welche Trafo der generalisierten Koord.

$F: \{q_k\} \rightarrow \{Q_k\}$ sind die Lagrange gl.

forminvariant?

Satz: Sei $F: \{q_k\} \rightarrow \{Q_k\}$ ein C^2 -Diffeomorphismus

d.h. umkehr eindeutige Abbildung,
 F und F^{-1} ex stetig diff. bar

$$\left(\begin{aligned} Q_i &= F_i(q_1, \dots, q_f, t) \\ q_k &= f_k(Q_1, \dots, Q_f, t) \end{aligned} \right. \text{ mit } \det \left(\frac{\partial f_k}{\partial Q_i} \right) \neq 0$$

Dann ist $\{Q_k(t)\}$ Lösung der Lagrange-Gln. zur transformierten Lagrange-Fkt. $\bar{L}(Q_1, \dots, \dot{Q}_1, \dots, t)$

$$\bar{L}(Q_k, \dot{Q}_k, t) := L(\underbrace{f_k(Q_i, t)}_{q_k}, \underbrace{\sum_i \frac{\partial f_k}{\partial \dot{Q}_i} \dot{Q}_i + \frac{\partial f_k}{\partial t}}_{\dot{q}_k}, t)$$

$\Leftrightarrow \{q_k(t)\}$ Lösung der Lagrange-Gln. in $L(q_k, \dot{q}_k, t)$

Beweis:

$$\begin{aligned} \frac{d}{dt} \frac{\partial \bar{L}}{\partial \dot{Q}_k} &= \sum_{l=1}^f \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_l} \frac{\partial \dot{q}_l}{\partial \dot{Q}_k} \right) = \sum_l \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_l} \frac{\partial q_l}{\partial Q_k} \right) \\ &= \sum_l \left\{ \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_l} \right) \right] \frac{\partial q_l}{\partial Q_k} + \frac{\partial L}{\partial \dot{q}_l} \frac{d}{dt} \left(\frac{\partial q_l}{\partial Q_k} \right) \right\} \\ &= \sum_l \left\{ \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_l} \right) \right] \frac{\partial q_l}{\partial Q_k} + \frac{\partial L}{\partial \dot{q}_l} \frac{\partial \dot{q}_l}{\partial Q_k} \right\} \end{aligned}$$

$$\frac{\partial \bar{L}}{\partial Q_k} = \sum_{l=1}^f \left[\frac{\partial L}{\partial q_l} \frac{\partial q_l}{\partial Q_k} + \frac{\partial L}{\partial \dot{q}_l} \frac{\partial \dot{q}_l}{\partial Q_k} \right]$$

$$\Rightarrow \frac{d}{dt} \frac{\partial \bar{L}}{\partial \dot{Q}_k} - \frac{\partial \bar{L}}{\partial Q_k} = \sum_{l=1}^f \frac{\partial q_l}{\partial Q_k} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_l} \right) - \frac{\partial L}{\partial \dot{q}_l} \right]$$



Variationsabl. ist „kovariant“ unter diffeomorphen Transformationen der generalisierten Koord.

Transformationsmatrix (nicht-singulär, d.h. $\det \neq 0$)

□

Es gibt unendl. viele äquivalente Sätze generalisierter Koordinaten!