English summary:

4.1 Legendre transform

$g(u) = x\frac{df}{dx} - f(x)$, applied to L and $\dot q$: $\dot q \frac{\partial L}{\partial \dot q} - L = \dot q\, p - L$

Hamilton function ("Hamiltonian"):

$$H(q_1, \dots, q_f, p_1, \dots, p_f, t) = \sum_k \dot q_k p_k - L(q_1, \dots, q_f, \dot q_1, \dots, \dot q_f, t)$$

4.2 Canonical equations of Hamilton

$$\left.\begin{aligned} \dot q_k &= \frac{\partial H}{\partial p_k} \\ \dot p_k &= -\frac{\partial H}{\partial q_k} \end{aligned}\right\}$$

$2f$ differential equations of 1^{st} order

(compare Lagrange's equations of 2^{nd} kind)

for scleronomous constraints, conservative forces:

$H = T + V$: total energy (conserved for temporal translational invariance)

Fortsetzung von Kapitel 4.2

Anwendungsschema des Hamilton-Formalismus:

(i) Wähle generalisierte Koordinaten $\underline{q} = (q_1, \dots, q_f)$

(ii) Transformation $\underline{r}_i = \underline{r}_i(q_1, \dots, q_f, t)$, $\dot{\underline{r}} = \dot{\underline{r}}(\underline{q}, \dot{\underline{q}}, t)$ $i = 1, \dots, N$

(iii) $L(\underline{q}, \dot{\underline{q}}, t) = T - V$ (konservative Kräfte)

(iv) generalisierte Impulse : $p_k = \frac{\partial L}{\partial \dot q_k}$ => $p_k = p_k(\underline{q}, \dot{\underline{q}}, t)$

Umkehrung $\Rightarrow \dot{q}_k = \dot{q}_k(\underline{q}, \underline{p}, t)$

(v) Legendre-Transformation: $H(\underline{q}, \underline{p}, t) = \sum_{k=1}^{f} \dot{q}_k p_k - L(\underline{q}, \dot{\underline{q}}, t)$

(vi) Kanonische Gleichungen: $\dot{q}_k = \frac{\partial H}{\partial p_k}$, $\dot{p}_k = -\frac{\partial H}{\partial q_k}$ aufstellen & integrieren

Beispiele:

(a) 1 Teilchen in Zylinderkoordinaten ohne Zwangsbedingungen

(i) $q_1 = r$, $q_2 = \varphi$, $q_3 = z$

(ii) $x = r\cos\varphi$, $\dot{x} = \dot{r}\cos\varphi - r\sin\varphi\,\dot{\varphi}$

$y = r\sin\varphi$, $\dot{y} = \dot{r}\sin\varphi + r\cos\varphi\,\dot{\varphi}$

(iii) $T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2)$, $V = V(r, \varphi, z)$

$\Rightarrow L(r, \varphi, z, \dot{r}, \dot{\varphi}, \dot{z}) = T - V$

(iv) generalisierte Impulse:

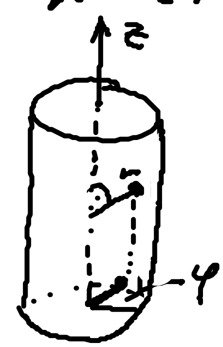
$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ Radialimpuls

$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi}$ z-Komponente des Drehimpulses (l_z)

$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$ z-Komponente des Impulses

(v) $H = p_r\dot{r} + p_\varphi\dot{\varphi} + p_z\dot{z} - L = \frac{1}{m}\left(p_r^2 + \frac{p_\varphi^2}{r^2} + p_z^2\right) - \frac{1}{2m}\left(p_r^2 + \frac{p_\varphi^2}{r^2} + p_z^2\right) + V$

$= \frac{1}{2m}\left(p_r^2 + \frac{p_\varphi^2}{r^2} + p_z^2\right) + V(r, \varphi, z)$



(vi) $\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$, $\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\varphi^2}{m r^3} - \frac{\partial V}{\partial r}$

$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{m r^2}$, $\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = -\frac{\partial V}{\partial \varphi}$

$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$, $\dot{p}_z = -\frac{\partial H}{\partial z} = -\frac{\partial V}{\partial z}$

spezielles Potenzial: $V(r)$ ($\hat{=}$ Zentralpotenzial bei ebener Bewegung)

$\dot{p}_r = \frac{p_\varphi^2}{m r^3} - \frac{\partial V}{\partial r}$

$$\left.\begin{array}{l} \dot{p}_\varphi = 0 \\ \dot{p}_z = 0 \end{array}\right\}$$ φ und z sind zyklische Variable

$\Rightarrow p_\varphi = \text{const.}$ (Drehimpulserhaltung)

$p_z = \text{const.} \overset{\text{oBdA}}{=} 0$ (ebene Bewegung)

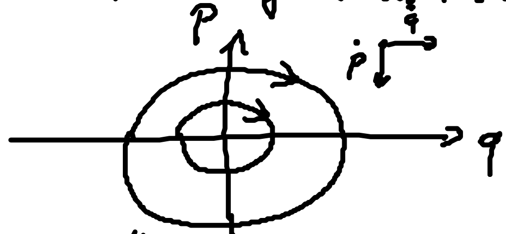
(b) 1-dimensionale harmonische Oszillator

$$L = \frac{m}{2}\left(\dot q^2 - \omega_0^2 q^2\right), \quad p = \frac{\partial L}{\partial \dot q} = m \dot q$$

$$H = \dot q p - L = \frac{p^2}{m} - \left[\frac{m}{2}\left(\frac{p^2}{m^2} - \omega_0^2 q^2\right)\right] = \frac{p^2}{2m} + \frac{m}{2}\omega_0^2 q^2$$

$\frac{\partial L}{\partial t} = 0$, skleronomes System $\Rightarrow$ Energieerhaltung $H = T + V = E$

$$\Rightarrow \frac{p^2}{2mE} + \frac{q^2}{\frac{2E}{m\omega_0^2}} = 1$$



Bahnkurven im Phasenraum: Ellipsen mit Halbachsen

$$a = \sqrt{\frac{2E}{m\omega_0^2}}, \quad b = \sqrt{2mE}$$

kanonische Gleichungen:

$$\left.\begin{aligned} \dot p &= -\frac{\partial H}{\partial q} = -m\omega_0^2 q \\ \dot q &= \frac{\partial H}{\partial p} = \frac{p}{m} \end{aligned}\right\} \mathrel{\hat{=}} \ddot q + \omega_0^2 q = 0$$

(definiert Richtungsfeld im Phasenraum)