English summary:

4.1 Legendre transform

$g(u) = x\frac{df}{dx} - f(x)$, applied to L and $\dot{q}$: $\dot{q}\frac{\partial L}{\partial \dot{q}} - L = \dot{q}\,p - L$

Hamilton function ("Hamiltonian"):

$$H(q_1,\dots,q_f,p_1,\dots,p_f,t) = \sum_k \dot{q}_k p_k - L(q_1,\dots,q_f,\dot{q}_1,\dots,\dot{q}_f,t)$$

4.2 Canonical equations of Hamilton

$$\left.\begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k} \\ \dot{p}_k &= -\frac{\partial H}{\partial q_k} \end{aligned}\right\}$$

2f differential equations of 1st order

(compare Lagrange's equations of 2nd kind)

for scleronomous constraints, conservative forces:

$H = T + V$: total energy (conserved for temporal translational invariance)

Fortsetzung von Kapitel 4.2

Anwendungsschema des Hamilton-Formalismus:

(i) Wähle generalisierte Koordinaten $\underline{q} = (q_1,\dots,q_f)$

(ii) Transformation $\underline{r}_i = \underline{r}_i(q_1,\dots,q_f,t)$, $\dot{\underline{r}} = \dot{\underline{r}}(\underline{q},\dot{\underline{q}},t)$ $\quad i=1,\dots,N$

(iii) $L(\underline{q},\dot{\underline{q}},t) = T - V$ (konservative Kräfte)

(iv) generalisierte Impulse: $p_k = \frac{\partial L}{\partial \dot{q}_k} \Rightarrow p_k = p_k(\underline{q},\dot{\underline{q}},t)$

Umkehrung $\Rightarrow \dot q_k = \dot q_k(\underline{q},\underline{p},t)$

(v) Legendre-Transformation: $H(\underline{q},\underline{p},t) = \sum_{k=1}^{f} \dot q_k p_k - L(\underline{q},\dot{\underline{q}},t)$

(vi) Kanonische Gleichungen: $\dot q_k = \frac{\partial H}{\partial p_k}$, $\dot p_k = -\frac{\partial H}{\partial q_k}$ aufstellen & integrieren

Beispiele:

(a) 1 Teilchen in Zylinderkoordinaten ohne Zwangsbedingungen

(i) $q_1 = r$, $q_2 = \varphi$, $q_3 = z$

(ii) $x = r\cos\varphi$, $\dot x = \dot r\cos\varphi - r\sin\varphi\,\dot\varphi$

$y = r\sin\varphi$, $\dot y = \dot r\sin\varphi + r\cos\varphi\,\dot\varphi$

(iii) $T = \frac{m}{2}\left(\dot r^2 + r^2\dot\varphi^2 + \dot z^2\right)$, $V = V(r,\varphi,z)$

$\Rightarrow L(r,\varphi,z,\dot r,\dot\varphi,\dot z) = T - V$

(iv) generalisierte Impulse:

$p_r = \frac{\partial L}{\partial \dot r} = m\dot r$ Radialimpuls

$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = m r^2\dot\varphi$ z-Komponente des Drehimpulses (ℓ_z)

$p_z = \frac{\partial L}{\partial \dot z} = m\dot z$ z-Komponente des Impulses

(v) $H = p_r\dot r + p_\varphi\dot\varphi + p_z\dot z - L = \frac{1}{m}\left(p_r^2 + \frac{p_\varphi^2}{r^2} + p_z^2\right) - \frac{1}{2m}\left(p_r^2 + \frac{p_\varphi^2}{r^2} + p_z^2\right) + V$

$= \frac{1}{2m}\left(p_r^2 + \frac{p_\varphi^2}{r^2} + p_z^2\right) + V(r,\varphi,z)$

(vi) $\dot r = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$, $\dot p_r = -\frac{\partial H}{\partial r} = \frac{p_\varphi^2}{m r^3} - \frac{\partial V}{\partial r}$

$\dot\varphi = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{m r^2}$, $\dot p_\varphi = -\frac{\partial H}{\partial \varphi} = -\frac{\partial V}{\partial \varphi}$

$\dot z = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$, $\dot p_z = -\frac{\partial H}{\partial z} = -\frac{\partial V}{\partial z}$

spezielles Potential: $V(r)$ ($\hat{=}$ Zentralpotential bei ebener Bewegung)

$\dot p_r = \frac{p_\varphi^2}{m r^3} - \frac{\partial V}{\partial r}$

$\left.\begin{array}{l} \dot p_\varphi = 0 \\ \dot p_z = 0 \end{array}\right\}$ φ und z sind zyklische Variable

$\Rightarrow p_\varphi = \text{const.}$ (Drehimpulserhaltung)

$p_z = \text{const.} \overset{\text{oBdA}}{=} 0$ (ebene Bewegung)

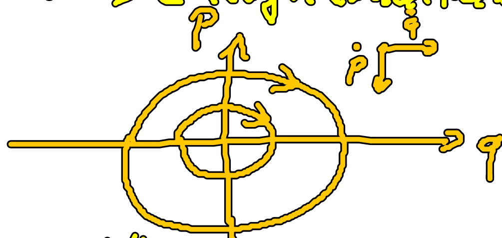
(b) 1-dimensionale harmonische Oszillator

$L = \frac{m}{2}(\dot{q}^2 - \omega_0^2 q^2)$, $p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$

$H = \dot{q}p - L = \frac{p^2}{m} - \left[\frac{m}{2}\left(\frac{p^2}{m^2} - \omega_0^2 q^2\right)\right] = \frac{p^2}{2m} + \frac{m}{2}\omega_0^2 q^2$

$\frac{\partial L}{\partial t} = 0$, skleronomes System $\Rightarrow$ Energieerhaltung $H = T + V = E$

$\Rightarrow \frac{p^2}{2mE} + \frac{q^2}{\frac{2E}{m\omega_0^2}} = 1$



Bahnkurven im Phasenraum: Ellipsen mit Halbachsen

$a = \sqrt{\frac{2E}{m\omega_0^2}}$, $b = \sqrt{2mE}$

kanonische Gleichungen: $\left.\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} = -m\omega_0^2 q \\ \dot{q} &= \frac{\partial H}{\partial p} = \frac{p}{m} \end{aligned}\right\} \hat{=} \ \ddot{q} + \omega_0^2 q = 0$

(definiert Richtungsfeld im Phasenraum)