English Summary:

Master eq. for one-step processes (birth-death): $W_{nn'} = \underset{\text{recomb.}}{r_{n'}\delta_{n,n'-1}} + \underset{\text{gen.}}{g_{n'}\delta_{n,n'+1}}$

$$\boxed{\dot{p}_n = r_{n+1} p_{n+1} + g_{n-1} p_{n-1} - (r_n + g_n) p_n}$$

stat. sol. $\boxed{p_n^* = p_0^* \prod_{n'=1}^{n} \frac{g_{n'-1}}{r_{n'}}}$

rate eqs. for $\langle N(t) \rangle = \sum_{n=0}^{\infty} n\, p_n(t)$

$\langle \dot{N} \rangle = \langle g_n \rangle - \langle r_n \rangle$; $\langle \dot{N} \rangle = -\gamma \langle N \rangle$ linear decay

2.2 Fokker-Planck-Gleichung

Zeitentwicklung eines kontinuierlichen Markov-Prozesses:

$X(t)$ (1-dim. Zufallsvar.)

$$\boxed{\frac{\partial}{\partial t} p(x,t|x_0,t_0) = \underbrace{-\frac{\partial}{\partial x}\left[A(x,t)\, p(x,t|x_0,t_0)\right]}_{\text{Drift}} + \underbrace{\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[B(x,t)\, p(x,t|x_0,t_0)\right]}_{\text{Diffusion}}}$$

Anfangsbed. $p(x,t_0|x_0,t_0) = \delta(x-x_0)$

$\hat{=}$ Ein-Zeit-Wahrsch. $p(x,t) = \int dx_0\, p(x,t;x_0,t_0) = \int dx_0\, p(x,t|x_0,t_0)\, p(x_0,t_0)$

mit Anf. bed. $p(x,t_0)$ (weniger singulär)

Randbed. (n-dim.)

Fokker-Planck (FP)-gl. ist lokale Bilanzgl.

$$\boxed{\frac{\partial p(x,t)}{\partial t} + \sum_i \frac{\partial}{\partial x_i} J_i(x,t) = 0}$$ $(\dot{p} + \operatorname{div} \underline{J} = 0)$

vgl. Teilchendichte ρ

$\dot{\rho} + \operatorname{div} \underline{J} = 0$

$\underline{J} = \underline{v}\rho - D\underline{\nabla}\rho$

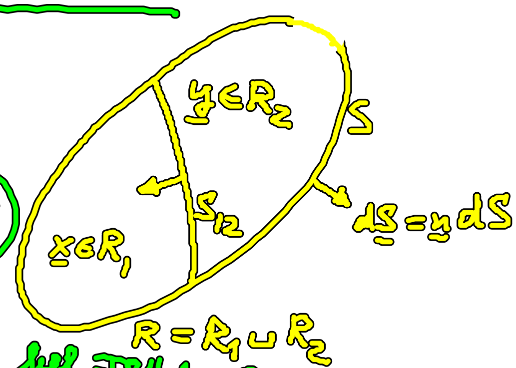
$\Rightarrow \dot{\rho} = -\rho \underline{\nabla}\cdot\underline{v} + D\Delta\rho$

mit Wahrscheinl.stromdichte

$J_i(\underline{x},t) = A_i(\underline{x},t)\, p(\underline{x},t) - \frac{1}{2}\sum_j \frac{\partial}{\partial x_j}\left(B_{ij}(\underline{x},t)\, p(\underline{x},t)\right)$

Globale Bilanzgl. für Gebiet $R \in \mathbb{R}^n$

$$P(R,t) := \int_R d^n x \; p(\underline{x},t)$$



$$\frac{\partial P}{\partial t} = -\int_S d\underline{S} \cdot \underline{J}(\underline{x},t) \qquad \left(\begin{array}{c}\text{Gauß'scher}\\ \text{Satz}\end{array}\right)$$

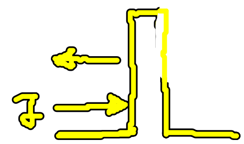
Netto-Wahrscheinl.-fluss durch bel. Fläche S_{12}:

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{R_1} d^n x \int_{R_2} d^n y \, \Big[\underbrace{p(\underline{x}, t+\Delta t; \underline{y}, t)}_{R_1 \leftarrow R_2} - \underbrace{p(\underline{y}, t+\Delta t; \underline{x}, t)}_{R_2 \leftarrow R_1} \Big] = \int_{S_{12}} d\underline{S} \cdot \underline{J}(\underline{x},t)$$

Randbed.:

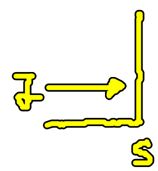
(a) <u>Reflektierende Barriere</u>:

$$\underline{n} \cdot \underline{J}(\underline{x},t)\Big|_{\underline{x} \in S} = 0$$



(b) <u>Absorbierende Barriere</u>:

$$p(\underline{x},t)\Big|_{\underline{x} \in S} = 0$$

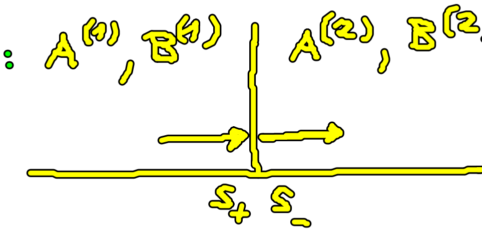


Teilchen aus dem System entfernt

(c) <u>Grenzfläche</u> zwischen 2 Medien: $A^{(1)}, B^{(1)}$ | $A^{(2)}, B^{(2)}$

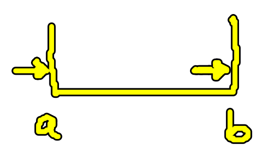
$$\underline{n} \cdot \underline{J}\Big|_{S_+} = \underline{n} \cdot \underline{J}\Big|_{S_-}$$

$$p\Big|_{S_+} = p\Big|_{S_-}$$



(d) <u>Periodische Randbed.</u>:

$$p(a,t) = p(b,t)$$
$$J(a,t) = J(b,t)$$



(e) <u>Natürliche Randbed.</u>: $A(a,t) = 0$

(Geschw. $= 0$, Rand wird nie erreicht)

Stationäre Lösung für homogenen Markov-Prozess

homogen $\Rightarrow$ A, B unabh. von t

1-dim. $\frac{d}{dx} J(x,t) = 0 \Rightarrow J(x) = \text{const.} = J(a) = J(b)$

(i) <u>reflekt. Randbed.</u> $\Rightarrow J(x) = 0$

$$\Rightarrow A(x)p^*(x) - \frac{1}{2}\frac{d}{dx}\left[B(x)p^*(x)\right] = 0 \quad \Rightarrow 2\frac{A}{B}dx = \frac{d(Bp^*)}{Bp^*}$$

$$\Rightarrow \boxed{p^*(x) = \frac{N}{B(x)} \exp\left[2\int_a^x dx' \frac{A(x')}{B(x')}\right]}$$

Potentiallösung, Normierung $\int_a^b dx\, p^*(x) = 1 \Rightarrow N$

(ii) <u>period. Randbed.</u> $\Rightarrow J(x) = J$

$$A(x)p^*(x) - \frac{1}{2}\frac{d}{dx}\left[B(x)p^*(x)\right] = J \quad (1) \text{ lin. inhom. Dgl.}$$

Mit $\psi(x) := \exp\left[2\int_a^x dx' \frac{A(x')}{B(x')}\right]$ ergibt sich

$$A\frac{\psi}{B} - \frac{1}{2}\frac{d}{dx}\psi = 0 \quad \left(\text{homog. Lös. } p^* = \frac{\psi}{B}\right)$$

$$\Rightarrow A = \frac{B}{2}\frac{\psi'}{\psi} \xrightarrow{\text{in } (1)} Bp^*\frac{\psi'}{\psi} - (Bp^*)' = 2J$$

$$\Rightarrow \frac{-(Bp^*)\psi' + (Bp^*)'\psi}{\psi^2} = -\frac{2J}{\psi}$$

$$\Rightarrow \left(\frac{Bp^*}{\psi}\right)' = -\frac{2J}{\psi}$$

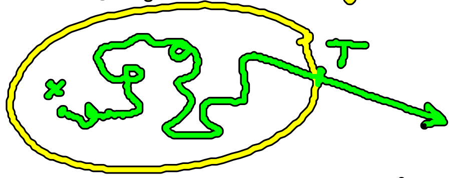
$$\overset{\int_a^x dx'}{\Longrightarrow} \left.\frac{Bp^*}{\psi}\right|_a^x = -2J\int_a^x \frac{dx'}{\psi(x')} \Rightarrow p^*(x)$$

Bestimmung von J durch period. Randbed. $p^*(a) = p^*(b)$

$$\Rightarrow p^*(x) = p^*(a)\,\frac{\int_a^x \frac{dx'}{\psi(x')}\frac{B(b)}{\psi(b)} + \int_x^b \frac{dx'}{\psi(x')}\frac{B(a)}{\psi(a)}}{\frac{B(x)}{\psi(x)}\int_a^b \frac{dx'}{\psi(x')}}$$

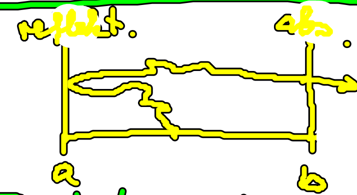
First passage time

Fragestellung: Wie lange hält sich ein Teilchen in einem vorgegebenen Gebiet auf?



Teilchen zwischen 1 absorb. und 1 reflektierenden Barriere

Ziel: Entweichzeit T (escape time)



Wahrsch., dass das Teilchen zur Zeit t noch in (a,b), wenn es bei x gestartet ist:

$$G(x,t) := \int_a^b dx'\, p(x',t\,|\,x,0) \equiv \mathrm{Prob}(T \geq t)$$

stationärer Prozess: $p(x',t\,|\,x,0) = p(x',0\,|\,x,-t)$

Rückwärts-FP-gl.: Rückwärtsentw. für $t' < t$ aus (x,t)

$$\frac{\partial p(x,t\,|\,y,t')}{\partial t'} = -A(y,t')\frac{\partial p(x,t\,|\,y,t')}{\partial y} - \frac{1}{2}B(y,t')\frac{\partial^2 p(x,t\,|\,y,t')}{\partial y^2}$$

homog. (A, B zeitunabh.):

$$\frac{\partial}{\partial t}G(x,t) = -\frac{\partial}{\partial t'}\int_a^b dx'\, p(x',0\,|\,x,\underbrace{t'}_{=-t})$$

$$\frac{\partial}{\partial t}G(x,t) = A(x)\frac{\partial}{\partial x}G(x,t) + \frac{1}{2}B(x)\frac{\partial^2}{\partial x^2}G(x,t)$$

statt Anf.bed. „End"bed.: $p(x',0\,|\,x,0) = \delta(x'-x)$

$$\Rightarrow G(x,0) = \begin{cases} 1 & a \leq x \leq b \\ 0 & \text{sonst} \end{cases}$$

Randbed.: 1 reflekt. + 1 absorb.

$$\partial_x G(a,t) = 0 \qquad G(b,t) = 0$$

(absorb. Randbed bedeutet: $\mathrm{Prob}(T \geq t) = 0$ wenn $x = b$
$\Leftrightarrow G(b,t) = 0$ (sofortige Absorption))

mittlere erste Übergangszeit (mean first passage time)

$$T(x) := \langle T \rangle = -\int_0^\infty t\, dG = -\int_0^\infty dt\, t \frac{\partial}{\partial t} G(x,t) \underset{\text{Int.}}{\overset{\text{part.}}{=}} \int_0^\infty dt\, G(x,t)$$

Dgl. für $T(x)$ aus der Rückwärts-FP-Gl.:

$$\int_0^\infty \frac{\partial}{\partial t} G(x,t)\, dt = \underbrace{G(x,\infty)}_{0} - \underbrace{G(x,0)}_{1} = -1$$

$$** \quad A(x) \frac{\partial}{\partial x} T(x) + \frac{1}{2} B(x) \frac{\partial^2}{\partial x^2} T(x) = -1$$

Randbed. $T'(a) = 0$ (refl.), $T(b) = 0$ (abs.)

T ausgedrückt durch Lösung der homog. Gl. $A\psi - \frac{1}{2} B \frac{\partial}{\partial x}\psi = 0$

$$\Leftrightarrow 2\frac{A}{B} dx = \frac{d\psi}{\psi}$$

$$\Leftrightarrow \boxed{\psi(x) = \exp\left[2\int_a^x dx' \frac{A(x')}{B(x')}\right]}$$

$\Rightarrow$ Lösung der inhom. Gl. ** $T(x)$ durch $\psi(x)$ ausgedrückt:

$$\boxed{T(x) = 2\int_x^b \frac{dy}{\psi(y)} \int_a^y dz \frac{\psi(z)}{B(z)}} \quad (*)$$

Beweis: $T' = -\frac{2}{\psi(x)} \int_a^x dz \frac{\psi(z)}{B(z)}$

$$T'' = -\frac{2}{\psi(x)^2}\left[\frac{\psi(x)^2}{B(x)} - \psi'(x)\int_a^x dz \frac{\psi(z)}{B(z)}\right]$$

$\psi' = \frac{2A}{B}\psi \Rightarrow$

$$\overset{(**)}{\Longrightarrow} AT' + \frac{1}{2}BT'' = -\frac{2A}{\psi}\int_a^x dz \frac{\psi}{B} - \left[1 - \overbrace{\frac{B}{\psi^2}\psi'}^{\frac{2A}{\psi}} \int_a^x dz \frac{\psi}{B}\right] \overset{!}{=} -1$$

Randbed.: $T'(a) = \int_0^\infty dt \frac{\partial}{\partial x} G(x,t)\Big|_{x=a} = 0$ ✓

$T(b) = 0$ ✓

□