Do. 30.6: EW203

6.6 Rotation

• Def. Rotation eines Vektorfeldes $\underline{a}(\underline{r})$

$$\operatorname{rot} \underline{a}(\underline{r}) = \underline{\nabla} \times \underline{a}(\underline{r}) \qquad (6.49)$$

.... Wirbelfeld von $\underline{a}(\underline{r})$

• kartesische Koordinaten:

$$\underline{\nabla} \overset{(6.26)}{=} \underline{e}_i \frac{\partial}{\partial x_i} \quad , \quad \underline{a}(\underline{r}) = a_i \underline{e}_i \qquad x_i, i = x, y, z$$

$$\longrightarrow \underline{\nabla} \times \underline{a}(\underline{r}) = \left(\underline{e}_i \frac{\partial}{\partial x_i}\right) \times \left(a_j(\underline{r}) \underline{e}_j\right)$$

$$\text{mit} \quad \underline{e}_i \times \underline{e}_j = \varepsilon_{ijk} \underline{e}_k \qquad (2.10)$$

$$\left[\frac{\partial}{\partial x_i} \underline{e}_j = 0\right] \quad = \frac{\partial}{\partial x_i} a_j(\underline{r}) \quad \varepsilon_{ijk} \underline{e}_k$$

$\varepsilon_{ijk} = \varepsilon_{kij}$

$\xrightarrow{\hspace{2em}}$

$i \to j$

$j \to k$

$k \to i$

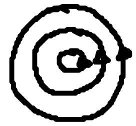
$$\boxed{\underline{\nabla} \times \underline{a}(\underline{r}) = \varepsilon_{ijk} \left(\frac{\partial}{\partial x_j} a_k\right) \underline{e}_i} \qquad (6.50)$$

$$[\underline{\nabla} \times \underline{a}(\underline{r})]_i = \varepsilon_{ijk} \left(\frac{\partial}{\partial x_j} a_k\right)$$

$$\left.\begin{aligned}
[\underline{\nabla} \times \underline{a}]_x &= \frac{\partial}{\partial y} a_z - \frac{\partial}{\partial z} a_y \\
[\underline{\nabla} \times \underline{a}]_y &= \frac{\partial}{\partial z} a_x - \frac{\partial}{\partial x} a_z \\
[\underline{\nabla} \times \underline{a}]_z &= \frac{\partial}{\partial x} a_y - \frac{\partial}{\partial y} a_x
\end{aligned}\right\} \qquad (6.51)$$

• Deutung I: rot $\underline{a}(\underline{r}) = \underline{\nabla} \times \underline{a}(\underline{r})$ identifiziert lokal Wirbel von Vektorfeldern (6.52)

Bsp. 1: $\underline{v} = \underline{\omega} \times \underline{r}$ (6.53) [s. Kap. 6.2]



... Prototyp eines Wirbels in Geschw.feldern

$\underline{\omega} = \frac{1}{2} \operatorname{rot} \underline{v}$ ... Wirbelstärke (6.54)

Beweis: $\operatorname{rot} \underline{v} = \underline{\nabla} \times (\underline{\omega} \times \underline{r})$

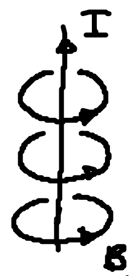
Hilfsformel: $\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a} \cdot \underline{c}) - \underline{c}(\underline{a} \cdot \underline{b})$

& $\underline{\nabla}$ $\underline{r}$

$$= \underline{\omega} \underbrace{(\underline{\nabla} \cdot \underline{r})}_{3} - \underbrace{(\underline{\omega} \cdot \underline{\nabla})\, \underline{r}}_{\omega_i \frac{\partial}{\partial x_i} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \underline{\omega}}$$

$$= 3\underline{\omega} - \underline{\omega} = 2\underline{\omega} \quad \text{qed.}$$

Bsp 2: Strom-Leiter:



$\operatorname{rot} \underline{B} \sim I$ (6.55)

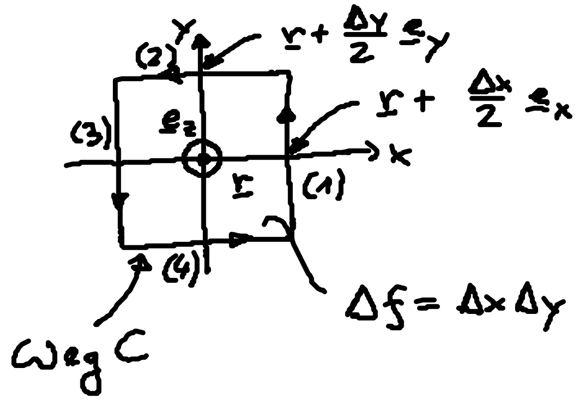
(Maxwell-Gl.!!!)

- Deutung II:

Ziel: Verdeutliche Maß für Wirbel von $\underline{a}(\underline{r})$ am Flächenelement Δf

(i) Orientierte Fläche = Flächenelement Δf mit Normalenvektor $\hat{\underline{\nu}}$ ($|\hat{\underline{\nu}}| = 1$) $\longrightarrow$ $\Delta \underline{f} = \Delta f\, \hat{\underline{\nu}}$

hier: $\Delta f = \Delta x\, \Delta y$, $\hat{\underline{\nu}} = \underline{e}_z$

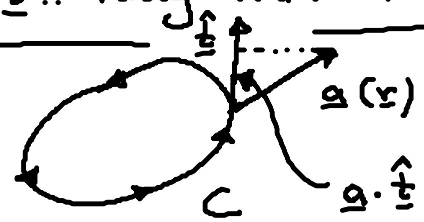


(ii) $\boxed{\text{Konvention: Wähle Weg } C \text{ um } \Delta f \text{ mit Rechter-Hand-Regel}}$

hier: $\hat{\underline{\nu}} = \underline{e}_z$, $C = (1) \to (2) \to (3) \to (4)$

(iii) Zirkulation / Wirbel:

$\boxed{\begin{array}{l}\text{Summiere Tangentialkomponente } \underline{a}\cdot\hat{\underline{t}} \\ \text{von } \underline{a}(\underline{r}) \text{ entlang } C \\ \underline{t} \,..\, \text{Tangentialvektor von } C\text{, orientiert wie } C\end{array}}$



hier: $$\Big[\underbrace{a_y\big(\underline{r} + \tfrac{\Delta x}{2}\underline{e}_x\big)}_{\underline{a}\cdot\underline{e}_y\ (1)} \underbrace{- a_y\big(\underline{r} - \tfrac{\Delta x}{2}\underline{e}_x\big)}_{\underline{a}\cdot(-\underline{e}_y)\ (3)}\Big]\Delta y$$

$$+ \Big[\underbrace{a_x\big(\underline{r} - \tfrac{\Delta y}{2}\underline{e}_y\big)}_{\underline{a}\cdot\underline{e}_x\ (4)} \underbrace{- a_x\big(\underline{r} + \tfrac{\Delta y}{2}\underline{e}_y\big)}_{\underline{a}\cdot(-\underline{e}_x)\ (2)}\Big]\Delta x$$

$$\overset{\text{Taylor}}{=} \Big[\cancel{a_y(\underline{r})} + \frac{\Delta x}{2}\frac{\partial}{\partial x}a_y - \cancel{a_y(\underline{r})} + \frac{\Delta x}{2}\frac{\partial}{\partial x}a_y\Big]\Delta y$$

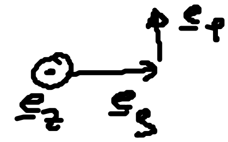
$$+ \Big[\cancel{a_x(\underline{r})} - \frac{\Delta y}{2}\frac{\partial}{\partial y}a_x - \cancel{a_x(\underline{r})} - \frac{\Delta y}{2}\frac{\partial}{\partial y}a_x\Big]\Delta x$$

$$= \Big(\frac{\partial}{\partial x}a_y - \frac{\partial}{\partial y}a_x\Big)\Delta x\,\Delta y = \underbrace{(\mathrm{rot}\,\underline{a})_z}_{(\mathrm{rot}\,\underline{a})\cdot\underline{e}_z}\ \underbrace{\Delta x\,\Delta y}_{\Delta f}$$

beliebige
Orientierung von $\Delta \underline{f}$

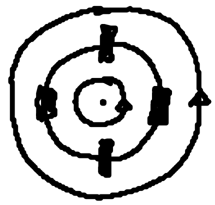
$$\boxed{\begin{array}{c}\text{Verwirbelung / Zirkulation um } \Delta \underline{f} \\ (\mathrm{rot}\,\underline{a})\cdot \Delta \underline{f} = (\mathrm{rot}\,\underline{a})\cdot \hat{\underline{\nu}}\,\Delta f\end{array}} \qquad (6.57)$$

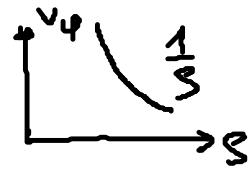
Bsp 3: $\underline{v}(\underline{r}) = \frac{g_0}{\varrho}\, \underline{e}_z \times \underline{e}_\varrho = \frac{g_0}{\varrho}\, \underline{e}_\varphi$ (6.58)



o.B. $\mathrm{rot}\,\underline{v} = 0$, $\underline{r} \neq 0$

Deutung:





- Zylinder-/Kugelkoord.: $\mathrm{rot}\,\underline{a}$ berechenbar!
- Regeln:

$$\left.\begin{array}{l}(1)\ \underline{\nabla}\times(\underline{a}+\underline{b}) = \underline{\nabla}\times\underline{a} + \underline{\nabla}\times\underline{b} \\ (2)\ \underline{\nabla}\times[f(\underline{r})\,\underline{a}] = f(\underline{r})\,\underline{\nabla}\times\underline{a} + [\underline{\nabla} f(\underline{r})]\times\underline{a}\end{array}\right\} \qquad (6.59)$$

Beweis: in kartesischen Koord.

- <u>wichtige Anwendungen</u>: Details s. Kursvorlesungen

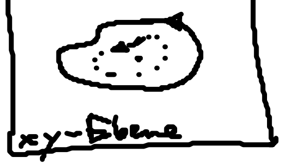
(1) Gradientenfelder sind wirbelfrei:

Satz:

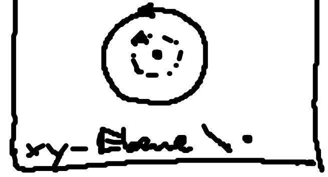
$$\boxed{\begin{array}{l}\text{Geg. } U(\underline{r}) \ldots \text{ Skalarfeld (2mal stetig diff.bar)} \\ \quad\ \ \underline{a}(\underline{r}) \ldots \text{ Vektorfeld} \\ \quad\ \ \text{im einfach zusammenhängenden Gebiet} \\ \text{dann: } \underline{a} = \mathrm{grad}\,U \longleftrightarrow \mathrm{rot}\,\underline{a} = 0\end{array}} \qquad (6.60)$$

(i) einfach zusammenhängend:
alle geschlossenen Kurven lassen sich auf Pkt. zusammenziehen

Bsp 1: ja



Bsp 2: nein



(ii) Beweis: $\longrightarrow$

$$[\underline{\nabla} \times (\underline{\nabla} U)]_x = \Big(\frac{\partial}{\partial y}\frac{\partial}{\partial z} - \underbrace{\frac{\partial}{\partial z}\frac{\partial}{\partial y}}_{\frac{\partial}{\partial y}\frac{\partial}{\partial z}}\Big) U(\underline{r}) = 0$$

$$[\ldots]_y = [\quad]_z = 0$$

$\longleftarrow$: hier nicht!

(iii) Mechanik: $\underline{F}(\underline{r}) = -\operatorname{grad} U$!!

(iv) Ausnahme: $U \sim \ln \varrho \longrightarrow \underline{a} \sim \frac{1}{\varrho}\underline{e}_\varrho$ mit $\operatorname{rot} \underline{a} = 0$

Vorsicht: nicht einfach zusammenhängend wegen $\varrho = 0$

(2) Satz: $$\boxed{\operatorname{div} \underline{B} = 0 \longleftrightarrow \underline{B} = \operatorname{rot} \underline{A}} \quad (6.61)$$

E-dynamik! Wirbelfelder sind quellenfrei!

(3) Laplace-Operator: $$\boxed{\underline{\nabla}^2 = \underline{\nabla}\cdot\underline{\nabla} = \Delta} \quad (6.62)$$

Bsp: Quellen eines Grad. feldes: $\underline{\nabla}\cdot(\underline{\nabla} U) = \underline{\nabla}^2 U$

(i) Potentialtheorie (E.-dynamik, Mechanik)

(ii) QM

kartes. Koord.: $$\boxed{\underline{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}} \quad (6.63)$$

(4) Hauptsatz der Vektoranalysis:

„Quellen & Wirbel bestimmen $\underline{a}(\underline{r})$ eindeutig"

$$\underline{a}(\underline{r}) = \underbrace{\underline{a}_t(\underline{r})}_{\substack{\operatorname{div} \underline{a}_t = 0 \\ \text{Wirbel} \\ \operatorname{rot}\underline{a} = \operatorname{rot}\underline{a}_t}} + \underbrace{\underline{a}_L(\underline{r})}_{\substack{\operatorname{rot} \underline{a}_L = 0 \\ \text{Quellen} \\ \operatorname{div}\underline{a} = \operatorname{div}\underline{a}_L}} + \underbrace{\underline{a}_R(\underline{r})}_{\substack{\operatorname{div}\underline{a}_R = \operatorname{rot}\underline{a}_R = 0 \\ \text{für Randbed.}}} \quad (6.64)$$

7. Integration von Feldern

- verschiedene Typen: Linien- / Flächen- / Volumenintegrale
  (1D) (2D) (3D)
- HM: genaue Def. / Berechnung
- hier: (1) anschauliche Definition
  (2) Auswahl von Integralen
  (3) physikal. Einsichten (oft) ohne Rechnung
  (4) Satz von Stokes & Gauß