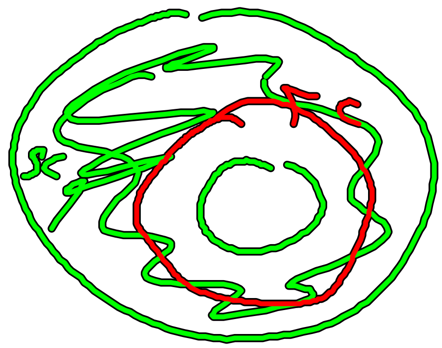
Superconductivity continued

$$\phi(\underline{r}) = \sqrt{n_s}\, e^{i\varphi(\underline{r})}$$

$$\underline{j}(\underline{r}) = \frac{ie^*}{2m^*}\left(\phi^*\underline{\nabla}\phi - \phi\nabla\phi^*\right) - \frac{e^*}{m^*}\underline{A}\phi^*\phi = 0$$

$$\frac{ie^*}{2m^*}\left[i\nabla\varphi\,\phi^*\phi + i\nabla\varphi\,\phi\phi^*\right] - \frac{e^*}{m^*}A\phi^*\phi = 0$$



$$\Leftrightarrow -\frac{e^*}{m^*}\phi^*\phi\left[\nabla\varphi + e^*\underline{A}\right] = 0$$

$$\Rightarrow \underline{A}(\underline{r}) = -\frac{\underline{\nabla}\varphi(\underline{r})}{e^*} \quad \text{(the phase matters)}$$

insert into flux equation

$$\Phi_B = -\oint_C d\underline{l}\,\frac{\nabla\varphi}{e^*} = \frac{\delta\varphi}{e^*}$$

phase change for round trip round contour

$\delta\varphi$ is given modulo 2π

$$\Rightarrow \delta\varphi = 2\pi\cdot p \qquad p = 0, 1, 2 \ldots$$

$$\Rightarrow \Phi_B = \frac{2\pi\hbar c}{e^*}\,p$$

$e^* = 2e$ found experimentally by Deaver and Fairbank in 1961

So we know now:

bosons with charge $2e$ give superconductivity

Ogg pairs

- in 1946 Ogg proposed electron pairing in real-space to yield bosons with $S=0$ or $S=1$
- later further developed by Schafroth, Butler and Blatt, <u>but</u>
  - no quantitative description of sc ($T_c \approx 10^4\,K$ much too large)
  - no microscopic force to explain attraction (electrons should repel each other)

$\Rightarrow$ real-space pairing does not work !

theory forgotten

Cooper pairs

- 1956 idea behind weak attraction by Cooper

  $\hookrightarrow$ consider
  - Fermi liquid (e.g. free electron metal)
  - two quasiparticles with $\underline{k}$ and $-\underline{k}$

two particle wave function

$$\Psi_0(\underline{r}_1, \underline{r}_2) = \sum_{\underline{k}} g_{\underline{k}}\, e^{i\underline{k}\underline{r}_1}\, e^{-i\underline{k}\underline{r}_2}$$

- assume spin singlet $\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$ $\underline{S} = 0$

(triplet is also possible
$\Rightarrow$ see unconventional superconductors)

ψ_0 symmetric $\Leftarrow$ antisymmetric

$$\psi_0(\underline{r}_1 - \underline{r}_2) = \sum_{k>k_F} g_{\underline{k}} \cos \underline{k}(\underline{r}_1 - \underline{r}_2)$$

into two particle SE

$$\left[\sum_{i=1,2} \frac{p_i^2}{2m} + V(\underline{r}_1, \underline{r}_2)\right]\psi_0 = E\psi_0$$

we get

$$(E - 2\epsilon_{\underline{k}})\, g_{\underline{k}} = \sum_{k>k_F} V_{\underline{k}\underline{k}'}\, g_{\underline{k}'}$$

unperturbed plane-wave energies

Fourier transform of V

$$V_{\underline{k}\underline{k}'} = \frac{1}{\Omega}\int d\underline{r}\, V(\underline{r})\, e^{i(\underline{k}'-\underline{k})\underline{r}}$$

Ω: volume; $\underline{r}$: $\underline{r}_1 - \underline{r}_2$

since $\epsilon_k > \epsilon_F$

if $E < 2\epsilon_F$ we could have a bound state

question? does a set of g_k exist such that $E < 2\epsilon_F$?

Coulomb potential :

$$V_{\underline{k}\underline{k}'} = V_{q=\underline{k}-\underline{k}'} = \frac{4\pi e^2}{q^2} > 0$$ not attractive

however: quasiparticles are no free electrons

$\Rightarrow$ screening

$\hookrightarrow$ Thomas-Fermi for simplicity

$$\varepsilon(\underline{k}) = \frac{\underline{k}^2 + \underline{k}_0^2}{\underline{k}^2} \neq 1$$

dielectric function

$$V(\underline{r}) = \frac{\varepsilon^{-1}(\underline{r}_1 - \underline{r}_2)}{|r_1 - r_2|}$$

$$\Rightarrow V(q) = \frac{4\pi e^2}{q^2 + \underline{k}_0^2}$$ still positive, but reduced

build on this idea: ions are positively charged

$\Rightarrow$ they respond to motion of electrons, too (although on timescale of phonons)

rough estimate (see Ashcroft Chapter 26)

$$V(q) = \frac{4\pi e^2}{q^2 + k_0^2} \frac{\omega^2}{\omega^2 - \omega_q^2}$$

$\omega = \frac{1}{\hbar}(\varepsilon_{\underline{k}} - \varepsilon_{\underline{k}'})$

ω_q : typical phonon frequency at q

if $\omega < \omega_q$ this (oversimplified) interaction could work ?

$\Rightarrow$ phonons could provide attraction

Cooper further simplified the potential

$$V_{\underline{k}\underline{k}'} = \begin{cases} -V & \text{for } |\epsilon_{\underline{k}} - \epsilon_{\underline{k}_F}|,\ |\epsilon_{\underline{k}'} - \epsilon_{\underline{k}_F}| < \overbrace{\hbar\omega_c}^{\text{average phonon energy}} \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad g_{\underline{k}} = V \frac{\sum g_{\underline{k}'}}{2\epsilon_{\underline{k}} - E}$$

apply $\sum_k$ and then cancel $\sum g_{\underline{k}}$

$$\Rightarrow \quad \frac{1}{V} = \sum_{\underline{k} > \underline{k}_F} \frac{1}{2\epsilon_k - E}$$

replace sum by integral

$$\frac{1}{V} = \int_{\epsilon_F}^{\epsilon_F + \hbar\omega_c} d\epsilon\, N(\epsilon) \frac{1}{2\epsilon - E} \overset{N(\epsilon) \approx N(\epsilon_F)}{=} \frac{1}{2} N(\epsilon_F) \ln \frac{2\epsilon_F - E + 2\hbar\omega_c}{2\epsilon_F - E}$$

$$\Leftrightarrow \quad E\left(e^{-\frac{2}{NV}} - 1\right) = 2\epsilon_F\left(e^{-\frac{2}{NV}} - 1\right) + 2\hbar\omega_c e^{-\frac{2}{NV}} \qquad N = N(\epsilon_F)$$

"weak coupling": $N(\epsilon_F) V \ll 1 \quad \Rightarrow \quad 1 - e^{-\frac{2}{N(\epsilon_F)V}} \approx 1$

$$E \approx \underbrace{2\epsilon_F}_{\text{energy for two independent electrons}} - 2\hbar\omega_c e^{-\frac{2}{N(\epsilon_F)V}} < 2\epsilon_F$$

$\Rightarrow$ electron pair is stable

important:

- binding outweighs kinetic energy gain for states above ϵ_F, regardless of size of V

- binding energy is not analytic at $V=0$
  $\Rightarrow$ perturbation theory does not apply
    $\Rightarrow$ that's why theory took so long to develop

- binding happens in $\underline{k}$-space, not in real-space (like Ogg tried it)

$$\psi_0 = \sum_{k>k_F} g_k \cos \underline{k}(\underline{r}_1 - \underline{r}_2)$$

$$= \sum_{k>k_F} \frac{V}{2(\epsilon_k - \epsilon_F) + 2\epsilon_F - E} \cos \underline{k}(\underline{r}_1 - \underline{r}_2)$$

Microscopic theory: BCS theory

we know our bosons, now find quantum mechanical description (ie. wavefunction, energy, etc.)

$$\Psi = \phi(\underline{r}_1\sigma_1, \underline{r}_2\sigma_2) \cdots \phi(\underline{r}_{N-1}\sigma_{N-1}, \underline{r}_N\sigma_N)$$

boson wfc, always the same

$$\psi_{BCS} = A\,\psi$$

antisymmetrize

put this into many-body SE and solve, maybe not too insightful

instead: the fast route: 2nd quantization

$$H = \sum_{\underline{k},\sigma} \underbrace{\xi_{\underline{k}}}_{\epsilon_k - \epsilon_F} \underbrace{c^+_{\underline{k}\sigma}}_{\text{creation op.}} c_{\underline{k}\sigma} + \sum_{\underline{k}\underline{k}'} c^+_{\underline{k}\uparrow} c^+_{-\underline{k}'\downarrow} V_{\underline{k}\underline{k}'} \underbrace{c_{-\underline{k}'\downarrow}}_{\text{annihilation op.}} c_{\underline{k}\uparrow}$$

then find ground state, so on so forth

- insted: mean field approximation

$$\sum_{\underline{k}'} V_{\underline{k}\underline{k}'} c_{-\underline{k}'\downarrow} c_{\underline{k}\uparrow} \approx -V\,\Theta(\hbar\omega_c - |\xi_{\underline{k}}|) \sum_{\underline{k}'} \Theta(\hbar\omega_c - |\xi_{\underline{k}'}|) \overline{(c_{-\underline{k}\downarrow} c_{\underline{k}\uparrow})}_{avg}$$

$$=: -\Delta_k = \begin{cases} -\Delta & \text{for } |\xi_{\underline{k}}| \leq \hbar\omega_c \\ 0 & \text{otherwise} \end{cases}$$

one more step:

we write $-c_{\underline{k}\downarrow} c_{\underline{k}\uparrow} = -c_{\underline{k}\downarrow} c_{\underline{k}\uparrow} + b_{\underline{k}} - b_{\underline{k}}$

$b_{\underline{k}} \to \langle c_{\underline{k}\downarrow} c_{\underline{k}\uparrow} \rangle_{avg}$

- insert into BCS Hamiltonian
- expand to first order
- apply mean-field approx. for V

$$\Rightarrow H_{MF} = \sum_{\underline{k}\sigma} \xi_{\underline{k}} c^+_{\underline{k}\sigma} c_{\underline{k}\sigma} - \sum_{\underline{k}} \left[\Delta_{\underline{k}} c^+_{\underline{k}\uparrow} c^+_{-\underline{k}\downarrow} + \Delta^*_k c_{-\underline{k}\downarrow} c_{\underline{k}\uparrow} - \Delta_k b^*_k \right]$$

this is an effective single-pair approximation
(i.e. only 2 c-operators not 4)

diagonalize MF Hamiltonian

- define new quasiparticles:

$$c_{\underline{k}\uparrow} = u_{\underline{k}} \alpha_{\underline{k}} + v_{\underline{k}} \beta_{\underline{k}}^+$$

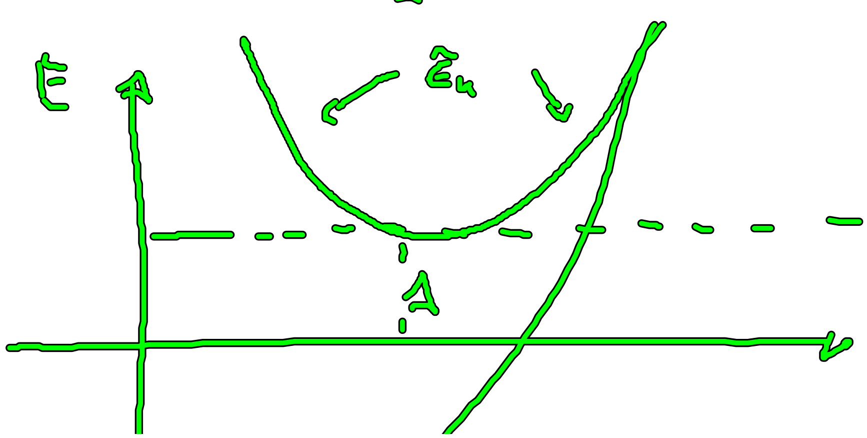
$$c_{-\underline{k}\downarrow} = u_{\underline{k}} \beta_{\underline{k}} - v_{\underline{k}} \alpha_{\underline{k}}^+$$

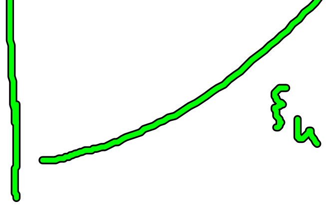
$\alpha_{\underline{k}}$ and $\beta_{\underline{k}}$ are new (rotated) quasiparticles

$$u_{\underline{k}}^2 = \frac{1}{2}\left(1 + \frac{\xi_{\underline{k}}}{\tilde{E}_k}\right) \qquad v_{\underline{k}}^2 = \frac{1}{2}\left(1 - \frac{\xi_{\underline{k}}}{\tilde{E}_{\underline{k}}}\right)$$

$$u_{\underline{k}} v_{\underline{k}} = -\frac{\Delta_k}{2\tilde{E}_k} \qquad \text{with} \quad \tilde{E}_k = \sqrt{\xi_{\underline{k}} + |\Delta_k|^2}$$

$$H_{MF} = E_0 + \sum_{\underline{k}} \tilde{E}_k \left(\alpha_{\underline{k}}^+ \alpha_{\underline{k}} + \beta_{\underline{k}}^+ \beta_{\underline{k}}\right)$$





Other consequence of energy gap

Einstein like heat capacity