English Summary:

## 3. Time-delayed feedback control

3.1 Delayed complex systems (delay differential equations)

3.2.1 Stabilization of unstable fixed points

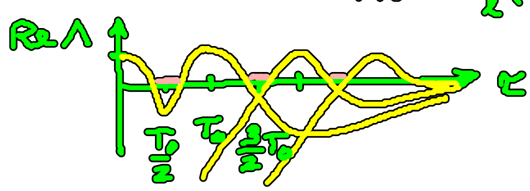
Normal form of unstable focus $\dot{z} = \underbrace{(\lambda \pm i\omega)}_{\text{eigenvalues of fixed point } z^*=0} z$, $z \in \mathbb{C}$, $\lambda > 0$

With time-delayed feedback control $\dot{z} = (\lambda \pm i\omega) z - K(z(t) - z(t-\tau))$

$\Rightarrow$ char. eq. ($z \sim e^{\Lambda t}$): $\boxed{\Lambda + K(1 - e^{-\Lambda\tau}) = \lambda \pm i\omega}$

solution of $\tilde{z} e^{\tilde{z}} = y$ : $\tilde{z} = W_\ell(y)$ Lambert fct. ($\ell \in \mathbb{Z}$)

$$\Lambda\tau = W_\ell\left(K\tau e^{-(\lambda \pm i\omega)\tau + K\tau}\right) + (\lambda \pm i\omega)\tau - K\tau$$




stabilization / destabilization for $\tau \approx \frac{2n+1}{2} T_0$, $T_0 = \frac{2\pi}{\omega}$

## Stabilisierung von instabilen Fixpunkten

char. Gl. $\boxed{\Lambda + K(1 - e^{-\Lambda\tau}) = \lambda \pm i\omega}$

Stabilitätsgrenzen: $\mathrm{Re}\,\Lambda = 0$

Char. Gl. in Re und Im aufgespalten mit $\Lambda = p + iq$:

$$\boxed{\begin{aligned} \lambda &= p + K\left[1 - e^{-p\tau}\cos q\tau\right] \\ \omega &= q + K e^{-p\tau} \sin q\tau \end{aligned}} \qquad \lambda > 0$$

$\mathrm{Re}\,\Lambda = p \overset{!}{=} 0$ :
$$\lambda = K(1 - \cos q\tau) \quad (1)$$
$$\omega = q + K \sin q\tau \quad (2)$$

Systempar. $\lambda, \omega$ geg.

Kontrollpar. $K, \tau > 0$

Kurvenpar. der Stab.grenze im $(K,\tau)$-Raum $q$

(1) $\Rightarrow \quad 0 \le 1 - \cos q\tau \le 2 \quad \Rightarrow \quad \frac{\lambda}{K} \le 2 \quad \Rightarrow \quad \boxed{K \ge \frac{\lambda}{2}}$

notwendige Bed. für Stabilisierung (minimale Rückkoppl.stärke)

$$K_{min} = \frac{\lambda}{2} \overset{(1)}{\Rightarrow} \cos q\tau = -1 \Rightarrow q\tau = (2n+1)\pi \quad (3)$$

$n = 0, 1, 2, ..$

$$\Rightarrow \sin q\tau = 0 \overset{(2)}{\Rightarrow} q = \omega$$

$$\Rightarrow \boxed{\tau = \frac{\pi}{\omega}(2n+1) = T_0 \frac{2n+1}{2}} \quad n = 0, 1, 2, ..$$

Für $\tau = \frac{2\pi n}{\omega} = nT_0$ ist keine Stabilisierung möglich, weil

$$(2) \Rightarrow q = \omega \overset{(1)}{\Rightarrow} \frac{K-\lambda}{K} = \cos(q\tau)\Big|_{q\tau = 2\pi n} = 1 \Leftrightarrow 1 - \frac{\lambda}{K} = 1 \quad ↯$$

<u>Analyt. Berechnung der Stabilitätsgrenze in der $(K,\tau)$-Ebene:</u>

$$(1),(2) \Rightarrow \left(\frac{K-\lambda}{K}\right)^2 + \left(\frac{\omega-q}{K}\right)^2 = \cos^2 q\tau + \sin^2 q\tau = 1$$

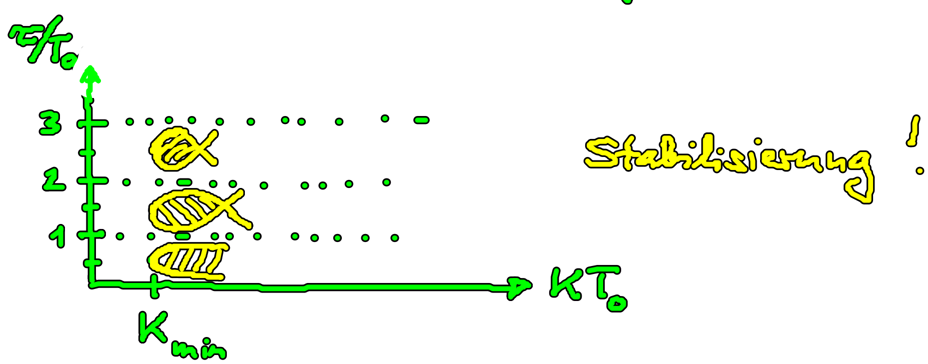
$$\Rightarrow \omega - q = \pm K \sqrt{1 - \left(\frac{K-\lambda}{K}\right)^2}$$

$$q = \omega \mp \sqrt{(2K-\lambda)\lambda}$$ eliminiert aus (1), (2)

feste Par. $\lambda, \omega$ des unkontrollierten Systems

$\Rightarrow$ Relation zwischen $\tau$ und $K$ aus (1): $\tau(K) = \dfrac{\arccos \frac{K-\lambda}{K}}{\omega \mp \sqrt{(2K-\lambda)\lambda}}$

arccos hat mehrere Zweige:



Hövel u. Schöll: Phys. Rev. E 72, 0620 (2005)

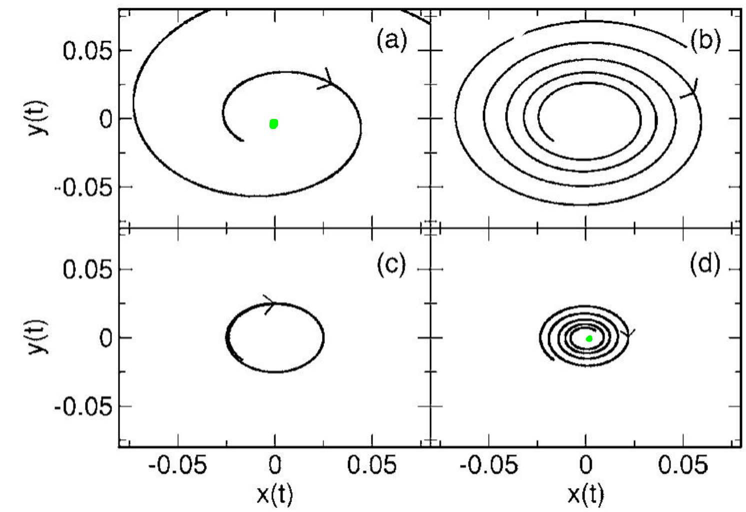



FIG. 1. Control of an unstable focus with $\lambda = 0.5$ and $\omega = \pi$ in the configuration space for different values of the feedback gain $K$. Panels (a)–(d) correspond to $K = 0, 0.2, 0.25$, and 0.3, respectively. The time delay $\tau$ of the TDAS control scheme is chosen as 1, corresponding to $\tau = T_0/2 = \pi/\omega$.

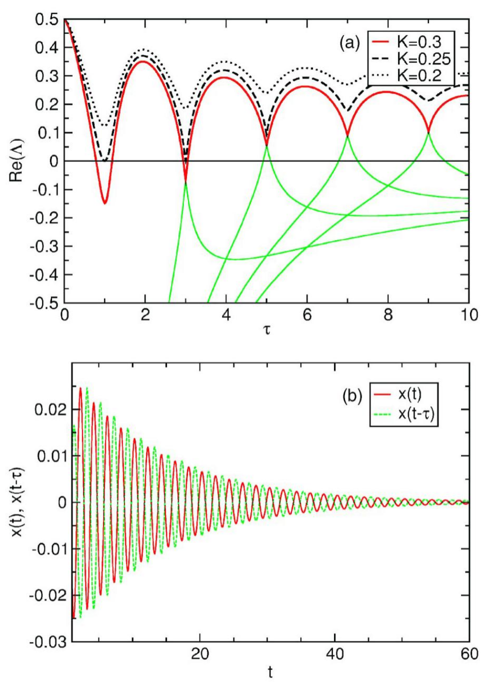
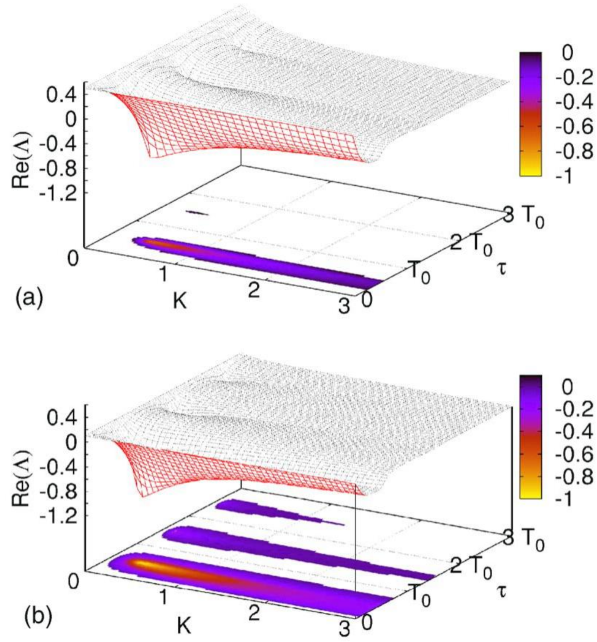



FIG. 2. (Color online) (a) Largest real part of the complex eigenvalues $\Lambda$ vs $\tau$ for $\lambda = 0.5$ and $\omega = \pi$ for different $K$. Some lower eigenvalues are also displayed for $K = 0.3$ (green online). (b) Time series of the $x$ component of the unstable focus: The solid line (red online) corresponds to $x(t)$, the dashed line (green online) to the delayed $x$ component $x(t-\tau)$ with $\tau = 1$. The parameters of the unstable focus and the control scheme are as in panel (d) of Fig. 1.

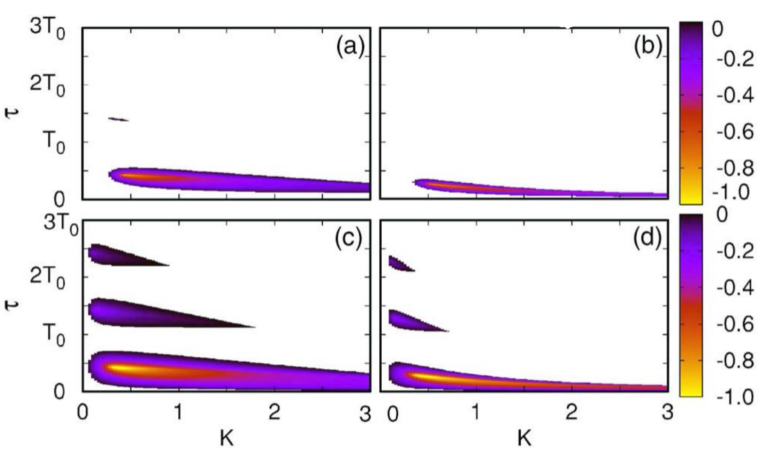
Stabilisierung für geeignete $\tau$ und $K$!
$\tau \approx \frac{T_0}{2}, \frac{3}{2} T_0, ..$




$\lambda = 0.5$

$\lambda = 0.1$

FIG. 3. (Color online) Domain of control in the $K$-$\tau$ plane and largest real part of the complex eigenvalues $\Lambda$ as a function of $K$ and $\tau$ according to Eq. (7). The two-dimensional projection at the bottom shows combinations of $\tau$ and $K$, for which $\mathrm{Re}(\Lambda)$ is negative and thus the control successful [panel (a): $\lambda=0.5$ and $\omega=\pi$; panel (b): $\lambda=0.1$ and $\omega=\pi$].




Latenzzeit $\delta$ :

$z(t-\delta) - z(t-\delta-\tau)$

FIG. 6. (Color online) Domain of control in the $K$-$\tau$ plane for different latency times [panels (a) and (c): $\delta=0.1$; panels (b) and (d): $\delta=0.3$]. The shaded areas indicate combinations of $\tau$ and $K$, for which the largest real part of the complex eigenvalues $\Lambda$ is negative and thus control is successful. The value of $\mathrm{Re}(\Lambda)$ is indicated by the greyscale (color online). The parameters of the unstable focus are chosen as $\omega=\pi$ in all panels and $\lambda=0.5$ in (a) and (b) and $\lambda$ $=0.1$ in (c) and (d).

Erweiterung (Socolar, Sukow, Gauthier, PRE 50, 3245 (94))

- multiple-time feedback (ETDAS, extended time-delay autosynchronization)

$$K \sum_{n=0}^{\infty} R^n [x(t-n\tau) - x(t-(n+1)\tau)]$$ Gedächtnispar. R ($0 \leq R < 1$)

Eigenwertgl.

$$\boxed{\Lambda + K \frac{1-e^{-\Lambda\tau}}{1-Re^{-\Lambda\tau}} = \lambda \pm i\omega}$$

Stab.bereich vergrößert

Dahms, Hövel, Schöll, PRE 76, 056201 (2007)

- Latenz-Effekte

$$K[x(t-\delta) - x(t-\delta-\tau)]$$ Latenzzeit $\delta$

Stab.bereich verkleinert

- phasen-abhängige Rückkoppl. $Ke^{i\varphi}$

$$K \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x(t)-x(t-\tau) \\ y(t)-y(t-\tau) \end{pmatrix}$$

- asymptot. Skalierungsverhalten für großes $\tau$

Yanchuk, Wolfrum, Hövel, Schöll, PRE 74, 026201 (2006)

Wolfrum et al., EPJ-ST 191, 91 (2010)

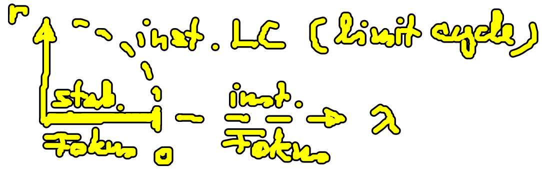
### 3.2.2 Stabilisierung instabiler periodischer Orbits

Normalform einer subkrit. Hopf-Bif.:

$$\boxed{\dot{z} = (\lambda + i\omega + (1+i\gamma)|z|^2)z + b(z(t-\tau) - z(t))}$$

$\lambda < 0$, $\omega = 1$, $\gamma > 0$, $b = b_0 e^{i\beta} \in \mathbb{C}$

ohne Kontrolle:




$z = re^{i\phi}$ ; $\dot r = (\lambda + r^2)\, r$

$\dot\phi = \omega + \gamma r^2$

LC: $r^2 = -\lambda$ ex. für $\lambda < 0$

$\dot\phi = \omega - \gamma\lambda \Rightarrow T = \frac{2\pi}{\omega - \gamma\lambda}$ Periode des UPO (unstable periodic orbit)

nichtinvasive Kontrolle (oBdA: $\omega = 1$):

wähle $\tau \overset{!}{=} nT = \frac{2\pi n}{1-\gamma\lambda}$, $n \in \mathbb{N}$

(Pyragas-Kurve in der $(\tau, \lambda)$-Ebene)

Odd-number orbit (= ungerade Zahl von reellen Floquet-Exp. > 0, hier $n = 1$; orbit ohne „Torsion")

Nakajima (1997): Stabilisierung von odd-number orbits durch zeitverzögerte Rückkopplung nicht möglich („Odd-number Theorem")

Fiedler, Flunkert, Georgi, Hövel, Schöll, PRL 98, 114101 (2007)

„Odd-number Theorem" gilt nicht!

Just, Fiedler, Georgi, Flunkert, Hövel, Schöll: PRE 76, 026210 (2007)

Gegenbeispiel: subkrit. Hopf-Bifurk. (Orbit ohne Torsion!)

Wähle $b_0$, $\beta$ geeignet!

Fiedler, Yanchuk, Flunkert, Hövel, Wünsche, Schöll: PRE 77, 066207 (2008)

Sattel-Knoten-Bif. von Grenzzyklen

Kehrt, Hövel, Flunkert, Dahlem, Rodin, Schöll: Eur. Phys. J B 68, 55 (2009)

raum-zeit. Muster

Erzgräber, Just: Physica D 238, 1680 (2009)

Brown, Postlethwaite, Silber: Physica D 240, 859 (2011)

Flunkert, Schöll: PRE 84, 016214 (2011)

Exp. mit el. Stromkreis: Loewenich, Benner, Just: PRE 82, 036204 (2010)

Laserexp.: Schikora et al., Henneberger, PRE 83, 026203 (2011)