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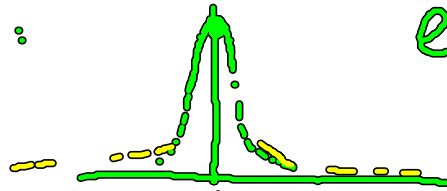
$$Z = \int \mathcal{D}\mathcal{H} e^{-S[\mathcal{H}]}$$

$$\int \frac{dH_1 \dots dH_N}{(2\pi)^{N/2} (\det \beta \mathbb{F})^{1/2}}$$

$$S[H_i] = \frac{1}{2\beta} \sum_{ij} H_i (\mathbb{F}^{-1})_{ij} H_j - \sum_i \ln 2 \cosh(H_i + \beta h_i)$$

Näherungsweise Berechnung des Integrals:

Betrachte Integranden:



Dominiert von der Umgebung
des Extremums von $f(x)$ bei

$$Z \approx e^{-S[\bar{H}_i]}, \quad \text{wobei}$$

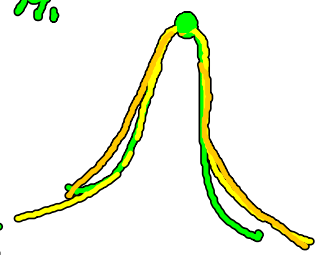
\bar{H}_i Extremum von $S[\bar{H}_i]$. „Nackte
Wäherung“

Nächste Näherung: Jetzt qualitativ um \bar{H}_i
erweitern

Also fordern

$$0 = \frac{\partial S}{\partial \bar{H}_i} = \frac{1}{2\beta} \sum_j \left[(\mathbb{F}^{-1})_{ij} \bar{H}_j + \bar{H}_j (\mathbb{F}^{-1})_{ji} \right]$$

$\mathbb{F}_{ij} = \mathbb{F}_{ji}$



$$0 = \frac{1}{2\beta} \sum_{ji} \left[\overbrace{J_{ik}(\sigma^{-1})_{ij} H_j}^{\delta_{kj}} + \overbrace{H_j(\sigma^{-1})_{ji} J_{ik}}^{\delta_{jk}} \right] - \sum_i J_{ik} \tanh(\bar{H}_i + \beta h_i) \delta_{jk}$$

$$0 = \frac{1}{\beta} \bar{H}_k - \sum_i J_{ik} \tanh(\bar{H}_i + \beta h_i)$$

$$\boxed{\frac{1}{\beta} \bar{H}_k = \sum_i J_{ik} \tanh(\bar{H}_i + \beta h_i)}$$

$$m_i = \langle \sigma_i \rangle = \frac{\frac{\partial Z}{\partial \beta h_i}}{Z} = \frac{1}{\beta} \frac{\partial \ln Z}{\partial h_i} = - \frac{\partial F}{\partial h_i} \quad \parallel e^{-\beta F} = Z \parallel$$

$$Z = \sum_{\sigma_i, \sigma_j} e^{\beta \sum_{ij} J_{ij} \sigma_i \sigma_j + \beta \sum_i h_i \sigma_i}$$

Führt in SP-Näherung $e^{-\beta F} \approx e^{-\beta S[H_i]}$

$$m_i = - \frac{\partial F}{\partial h_i} \mapsto - \frac{\partial S[H_i]}{\partial \beta h_i}$$

$$S = \frac{1}{2\beta} \sum_{ij} H_i (\sigma^{-1})_{ij} H_j - \sum_i \ln Z \cosh(H_i + \beta h_i)$$

$$m_i = - \underbrace{\frac{\partial S[\bar{H}_i]}{\partial \bar{H}_i}}_0 \frac{\partial \bar{H}_i}{\partial \beta h_i} + \frac{\partial}{\partial \beta h_i} \sum_i \ln 2 \cosh(\bar{H}_i + \beta h_i)$$

Extremaleigenschaft

$$= \tanh(\bar{H}_i + \beta h_i)$$

Jetzt

$$\frac{1}{\beta} \bar{H}_k = \sum_j \mathcal{J}_{jk} \tanh(\bar{H}_j + \beta h_j)$$

$$m_i = \tanh\left(\beta \sum_j \mathcal{J}_{ji} m_j + \beta h_i\right)$$

Also:

$$m_i = \tanh\left(\beta \sum_j \mathcal{J}_{ji} m_j + \beta h_i\right)$$

- Das ist die Mean-Field-Näherung
- Sattelpunktnäherung $\hat{=}$ MF
- Quadratische Fluktuationen geben Verbesserung des MF-Ergebnisses
 \Rightarrow systematische Entwicklung um das MF-Theorem herum
 „loop-wise“ expansion

$$F = U - TS$$

$$= \langle H \rangle - TS'$$

Stat. Operator $\rho = \frac{e^{-\beta H}}{\text{Tr } e^{-\beta H}}$

J Gibbs: Variationsprinzip

$$\begin{aligned} F &\leq \langle H - TS \rangle = \\ &= \text{Tr } \rho_0 (H + T k_B \ln \rho_0) \\ &= \underbrace{\text{Tr } \rho_0 H}_{\langle H \rangle_0} + k_B T \text{Tr } \rho_0 \ln \rho_0 \end{aligned}$$

wobei ρ_0 ein beliebiger GG-Zustand
zur Temperatur T ist

F ist minimal im Vergleich zu allen anderen
GG Zuständen ρ_0 ,

$$" = " \text{ für } \rho_0 = \rho.$$

(Gibbs - Bogoliubov - Ungleichung)

Beweis: R.P. Feynman „Statistical Physics“.

Anwender für $\mathcal{H} = -\frac{1}{2} \sum_{ij} J_{ij} \sigma_i \sigma_j - h \sum \sigma_i$

gibt "Test-Operator"

$$S_B \equiv \frac{1}{Z_B} e^{\beta B \sum \sigma_i}$$

↑ Parameter

$$Z_B = N \quad 2 \cosh \beta B$$

• $\langle H \rangle$, hier für:

$$\langle \sigma_i \rangle = \frac{\sum_{\sigma_1 \dots \sigma_N} \sigma_i e^{\beta B [\sigma_1 + \dots + \sigma_N]}}{\sum_{\sigma_1 \dots \sigma_N} e^{\beta B [\sigma_1 + \dots + \sigma_N]}}$$

$$\langle \sigma_i \sigma_j \rangle = \frac{\sum_{\sigma_1 \dots \sigma_N} \sigma_i \sigma_j e^{\beta B [\sigma_1 + \dots + \sigma_N]}}{\sum_{\sigma_1 \dots \sigma_N} e^{\beta B [\sigma_1 + \dots + \sigma_N]}}$$

$i \neq j$

$$= \tanh^2 \beta B$$

$i=j$ folgt mittels $J_{ii} = 0$

Damit $\langle H \rangle = -\frac{1}{2} \sum_{ij} J_{ij} \tanh^2 \beta B - N h \tanh \beta B$

homogen:

$$= \frac{1}{2} \sum_{ij} J_{ij} = \frac{N}{2} \sum_j J_{ij} = \frac{N}{2} J$$

$$\langle H \rangle = -\frac{N}{2} J \tanh^2 \beta B - N h \tanh \beta B$$

$$\langle S \rangle = - \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \text{Tr} \rho_B \ln \rho_B = - \beta B \langle \sum_i \sigma_i \rangle$$

$$= - \beta B N \frac{e^{+\beta B} \sum \sigma_i}{Z_B} + \ln Z_B$$

$$= - \beta B N \tanh \beta B + \ln 2 \cosh \beta B$$

Insgesamt

$$F \llcorner \langle H - TS \rangle \equiv \mathcal{F}(B)$$

$$\mathcal{F}(B) \equiv - \frac{N}{2} J \tanh^2 \beta B +$$

$$+ N(B-h) \tanh \beta B$$

$$- TN \ln(2 \cosh \beta B)$$

Jetzt minimieren!

$$\mathcal{F}'(B) = -N\beta J \tanh \beta B [1 - \tanh^2 \beta B]$$

$$+ N \tanh \beta B + N(B-h)\beta [1 - \tanh^2 \beta B]$$

$$- N \tanh \beta B$$

$$= N\beta (B-h - J \tanh \beta B) [1 - \tanh^2 \beta B]$$

$$= 0 \qquad = 0$$

$$\tanh' x = 1 - \tanh^2 x$$

Selbstkonsistenzgleichung

$$\left. \begin{aligned} B-h &= J \tanh \beta B \\ m &= \tanh \beta B \end{aligned} \right\} \begin{aligned} B-h &= Jm \\ B &= Jm + h \end{aligned}$$

$$\Rightarrow m = \tanh \beta (Jm + h)$$

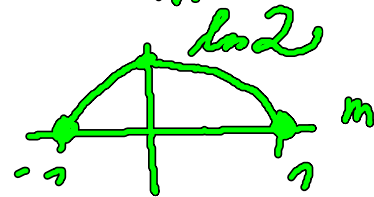
Betrachte nochmals

$$\frac{\langle H \rangle}{N} = -\frac{J}{2} m^2 - hm$$

$$\frac{\langle S \rangle}{N} = -\sum_{\pm} \frac{1 \pm m}{2} \ln \frac{1 \pm m}{2}$$

(S/N) Shannon Form

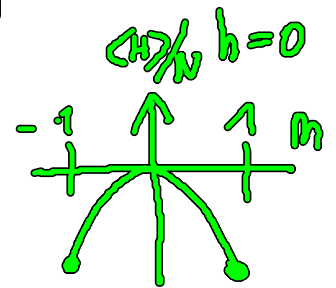
$$F < \langle H \rangle - T \langle S \rangle$$



Entwicklung in der Nähe

des kritischen Punktes: $|m| \ll 1$

$$\begin{aligned} \frac{\langle H-TS \rangle}{N} &= -\frac{J}{2} m^2 - hm + m(Jm + h) \\ &\quad - T \ln 2 - \frac{T}{2} (m^2 + \frac{1}{2} m^4 + \dots) \end{aligned}$$



[Ans: $\frac{\langle S \rangle}{N} = -\beta B \tanh \beta B + k_B \ln 2 \cosh \beta B$

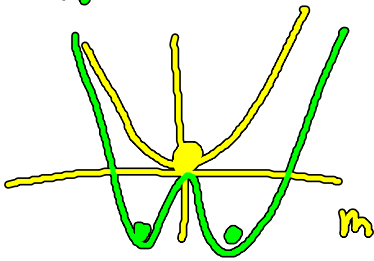
$$= -\beta (\mathcal{J}m+h) m + k_B \ln 2 + k_B \ln \frac{1}{\sqrt{1 - \tanh^2 \beta B}}$$

$$= -\beta (\mathcal{J}m+h) m + k_B \ln 2 - \frac{1}{2} k_B \ln (1 - m^2) m^2$$

$\underbrace{\hspace{10em}}_{-m^2 - m^4 - \dots}$

$$\frac{\langle H - TS \rangle}{N} = -T \left[\ln 2 + \frac{1}{2} \left(1 - \frac{T_c}{T} \right) m^2 + \frac{m^4}{4} + \dots \right]$$

$\underbrace{\hspace{10em}}_{O(m^6)}$



$$k_B T_c = \mathcal{J}$$