• 4.3 a) freier Fall mit Stokesscher Reibung

$$\tau = \frac{m}{6\pi\eta a} \qquad \left.\begin{aligned} &\eta(H_2O) = 10^{-3}\,\frac{kg}{ms} \\ &a = 10^{-6}\,m \\ &\rho = 1\,\frac{g}{cm^3} = 10^3\,\frac{kg}{m^3} \end{aligned}\right\} \; m = \frac{4\pi}{3} a^3 \rho \approx 4 \cdot 10^{-15}\,kg$$

$$\longrightarrow \underline{\tau \approx 10^{-7}\,s}$$

5. Eindimensionaler harmonischer Oszillator

• volles Problem: $m\ddot{x} = -fx - \alpha\dot{x} + F(t)$ (5.1)

5.1 Ungedämpfte freie Bewegung

a) Lösung: $\boxed{m\ddot{x} + fx = 0}$ (5.3)

• Lsg:

$$\boxed{\begin{aligned} & x(t) = a\cos\omega_0 t + b\sin\omega_0 t, \qquad \omega_0^2 = \frac{f}{m} \quad \text{... Eigenfrequenz} \\ & A^2 = a^2 + b^2,\ \tan\varphi = \frac{b}{a} \; \left\{ \begin{aligned} a &= A\cos\varphi \\ b &= A\sin\varphi \end{aligned} \right. \;\Big| = A\cos(\omega_0 t - \varphi) \\ & \left.\begin{aligned} &A \text{ ... Amplitude} \\ &\varphi \text{ ... Phase} \end{aligned}\right\} \text{der Schwingung} \end{aligned}} \quad (5.4)$$

b) komplexe Darstellung: $z = x + iy \in \mathbb{C}$ (5.5) $i = \sqrt{-1}$

$$\left.\begin{aligned} \ddot{x} + \omega_0^2 x &= 0 \\ \ddot{y} + \omega_0^2 y &= 0 \end{aligned}\right\} \xrightarrow[\text{Linearität}]{} \ddot{z} + \omega_0^2 z = 0 \quad (5.6)$$

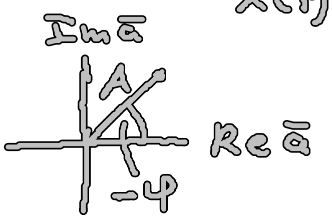
• sys. Lsg. durch Ansatz: $z(t) = e^{\lambda t}$ in (5.6)

$$\lambda^2 \cancel{e^{\lambda t}} + \omega_0^2 \cancel{e^{\lambda t}} = 0 \longrightarrow \boxed{\begin{aligned} \lambda^2 &= -\omega_0^2 \\ \lambda &= \pm i\omega_0 \end{aligned}} \quad (5.7) \quad \text{..., „charakt. Polynom"}$$

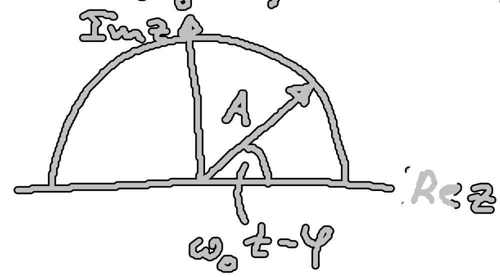
$[\sqrt{-1} = i]$

$\longrightarrow \quad z(t) = \bar{a}\, e^{i\omega_0 t} + \bar{b}\, e^{-i\omega_0 t}\,, \quad \bar{a}, \bar{b} \in \mathbb{C}$

$$x(t) = \operatorname{Re} z(t) \underset{\text{o.B.d.A}}{=\!=\!=} \operatorname{Re}\big[A\, \underbrace{e^{i(\omega_0 t - \varphi)}}_{\cos(\omega_0 t-\varphi) + i \sin(\omega_0 t - \varphi)}\big] \qquad (S.8)$$



(i) $\bar{a} = A e^{-i\varphi}$
$\bar{b} = 0$

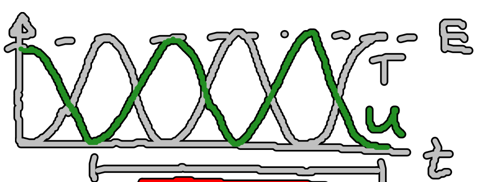


(ii) i.a. bestimme $\bar{a}, \bar{b}$ so, daß Anfangs-bed. erfüllt sind

c) <u>Energiebilanz</u>: $(\varphi = 0 \longrightarrow x(t) = A \cos\omega_0 t)$

- $E = T + U$ mit $U = \frac{f}{2} x^2 \longrightarrow -\frac{\partial U}{\partial x} = -fx \quad , \; f = m\omega_0^2$

$$\boxed{\left.\begin{aligned} T &= \frac{m}{2}\dot{x}^2 = \frac{m}{2}\omega_0^2 A^2 \sin^2\omega_0 t \\ U &= \frac{m}{2}\omega_0^2 x^2 = \frac{m}{2}\omega_0^2 A^2 \cos^2\omega_0 t \end{aligned}\right\} E = \frac{m}{2}\omega_0^2 A^2} \qquad (S.9)$$



$\boxed{t_0 = \frac{2\pi}{\omega_0}}$ ... Periodendauer (S.10)

- Mittelwerte:

Trick

$$\langle T \rangle = \frac{1}{t_0}\int_0^{t_0} dt\, T(t) \underset{(S.9)}{=\!=} E\, \underbrace{\frac{1}{\omega_0 t_0}\int_0^{2\pi} d(\omega_0 t)\, \sin^2\omega_0 t}_{\frac{1}{2}} = \frac{E}{2}$$

$$\langle U \rangle = \frac{1}{t_0}\int_0^{t_0} dt\, U(t) = \frac{E}{2} \qquad (S.11)$$

... Gleichverteilung im Mittel!

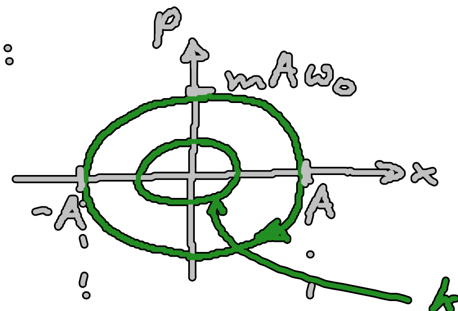
[NB. Trick $\frac{1}{2\pi}\int_0^{2\pi} \sin^2 x\, dx = \frac{1}{2}\,\frac{1}{2\pi}\int_0^{2\pi} \underbrace{(\sin^2 x + \cos^2 x)}_{1} dx = \frac{1}{2}$]

d) **Bewegung im Phasenraum:** $\boxed{x\text{-}p\text{-Ebene}}$

- p konjugierte Koord. zu x
  $\hookrightarrow$ $[px] = [Et] = Js$ ... Wirkung

- hier: $\left.\begin{array}{l} x(t) = A\cos(\omega_0 t) \\ p(t) = m\dot{x}(t) = -m A\omega_0 \sin(\omega_0 t) \end{array}\right\}$ Parameterdarstellung einer Ellipse

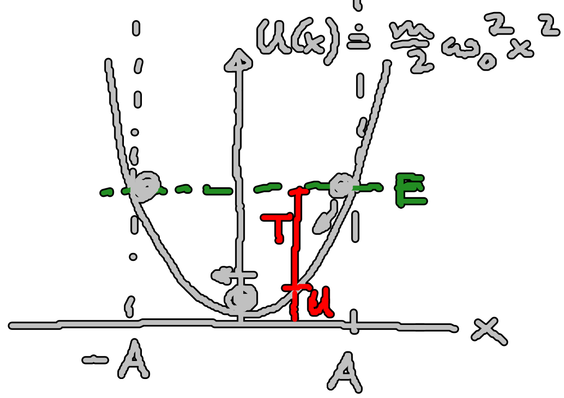
- Phasenporträt:



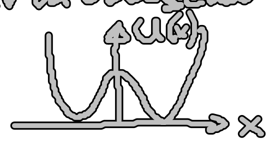
kleineres A, E

- period. Bewegung
  $\longrightarrow$ geschlossene Kurve
- Umlaufzeit: $t_0 = \frac{2\pi}{\omega_0}$

- vgl:



- Anharmonischer Oszillator: qualitatives Phasenporträt $\rightarrow$ Üb.



- Wirkung einer Bewegung:

$$\boxed{S := \oint \underbrace{p(x)\,dx}_{\text{eingeschlossene Fläche im Phasenraum}} = \int_0^{t_0} p(x)\,\dot{x}\,dt} \qquad (5.12)$$

ein Umlauf

- harm. Oszillator: $S = \underbrace{\pi m\omega_0 A^2}_{\text{Fläche Ellipse}} \underset{(5.3)}{=} 2\pi\frac{E}{\omega_0} = E t_0!$ (5.13)

- Klassik: S kontinuierlich

QM: $S_n = 2\pi\hbar\left(n+\frac{1}{2}\right)$, $n = 0, 1, \ldots\,!$

$\xrightarrow[(5.13)]{E = S/t_0}$ $\underbrace{E_n = \hbar\omega_0\left(n+\frac{1}{2}\right)}_{\text{Plancksche Quantenhypothese}}$ (5.14)

Bohr: S_n auch für e^- im Atom!

5.2 Gedämpfter, freier harm. Oszillator

· (5.1) $\longrightarrow$ $m\ddot{x} + \underbrace{\alpha\dot{x}}_{\substack{\text{nicht}\\\text{konserv.}\\\text{System}}} + fx = 0 \xrightarrow{x = \mathrm{Re}\,z}$ $\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = 0$ (5.16)

$\boxed{\begin{aligned} f &= m\omega_0^2 \quad (5.4) \\ \gamma &= \frac{\alpha}{2m} \quad (5.15) \end{aligned}}$ $[\gamma] = \frac{1}{s}$!

· Lsg.ansatz: $z(t) \sim e^{\lambda t}$ in (5.16)

$\longrightarrow$ charakt. Polynom: $\lambda^2 + 2\gamma\lambda + \omega_0^2 = 0$

$\longrightarrow$ $\boxed{\lambda_{1/2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}}$ (5.17)

allg. Lsg: $z(t) = \bar{a}\,e^{\lambda_1 t} + \bar{b}\,e^{\lambda_2 t}$, $\bar{a}, \bar{b} \in \mathbb{C}$ (5.18)

· Erinnerung: Energiebilanz (2.25): $\frac{d}{dt}(T+U) = F_{diss}\,\dot{x} = -\alpha\dot{x}^2$

· Fallunterscheidung

a) geringe Dämpfung: $\gamma < \omega_0$

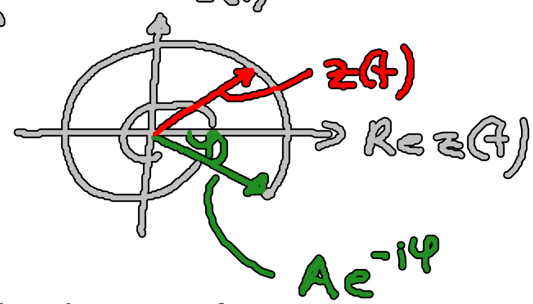
$\sqrt{-1} = i \longrightarrow$ $\boxed{\lambda_{1/2} = -\gamma \pm i\omega_d\,, \quad \omega_d = \sqrt{\omega_0^2 - \gamma^2} < \omega_0}$ (5.19)

(S.19) ⟶ reelle Lsg:

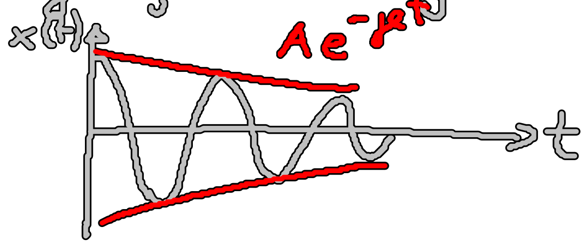
$$\boxed{\begin{aligned} x(t) &= \mathrm{Re}\, z(t) = \mathrm{Re}\left[A e^{-\gamma t} e^{i(\omega_d t - \varphi)}\right] \\ &\quad \bar{a} = A e^{-i\varphi} \\ &\quad \bar{b} = 0 \\ &= \underbrace{A e^{-\gamma t}}_{\text{„gedämpfte Amplitude"}} \underbrace{\cos(\omega_d t - \varphi)}_{\text{period. Schw.}} \end{aligned}}$$

(S.20)

- Zeigerdiagramm für komplexe Lsg:



- reelle Lsg = frei abklingende Schwingung mit ω_d:



<u>2 Kenngrößen</u>: 1. Relaxationszeit/Abklingzeit: $\boxed{\tau = \frac{1}{\gamma} = \frac{2m}{\alpha}}$ (S.15) (S.21)

2. Oszillatorgüte:

$$\boxed{Q = 2\pi \frac{\text{mittlere Energie}: \frac{\omega_d}{2\pi}\int_0^{2\pi/\omega_d} (T+U)\, dt}{\text{Energieverlust pro Periode}: \alpha \int_0^{2\pi/\omega_d} \dot{x}^2\, dt}}$$ (S.22)

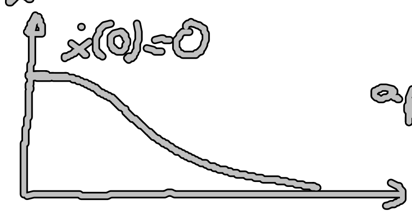
hohe Güte: $\tau \gg \frac{1}{\omega_0}$: $Q = 2\pi \dfrac{\frac{1}{2} m \omega_0^2 A^2 e^{-2\gamma t}}{\alpha A^2 e^{-2\gamma t} \omega_d^2 \underbrace{\int_0^{2\pi/\omega_d} \cos^2(\omega_d t - \varphi)\, dt}_{\pi/\omega_d}}$

$\omega_d \approx \omega_0 \longrightarrow$ $$\boxed{Q = \frac{m\omega_0}{\alpha} = \frac{\omega_0 \tau}{2} \gg 1}$$ (S.23)

b) <u>starke Dämpfung</u>: $\gamma > \omega_0 \longrightarrow \lambda_{1/2} = -\gamma \pm \sqrt{\gamma^2-\omega_0^2} \in \mathbb{R}$

$\longrightarrow$ rein exp. Abfall: $\boxed{x(t) = a e^{-(\gamma-\sqrt{\dots})t} + b e^{-(\gamma+\sqrt{\dots})t}}$

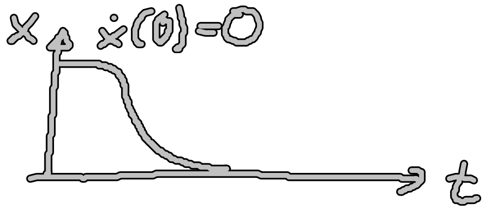
$a, b \in \mathbb{R}$



aperiodische Dämpfung
Kriechbewegung

c) <u>aperiodischer Grenzfall</u>: $\gamma = \omega_0 \longrightarrow \underbrace{\lambda_{1/2} = -\gamma}_{\text{Lsg. entartet}}$

$\xrightarrow{\text{„Mathe"}}$ $\boxed{x(t) = a e^{-\gamma t} + b t e^{-\gamma t}}$ (5.25)



- Anwendung: ideal für Messgeräte
  ideale Dämpfung, klingt am schnellsten ab: $\gamma > \gamma - \sqrt{\dots}$
  $\longrightarrow$ Zeiger geht am schnellsten gegen Messwert